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Lectures on the Bose-Einstein Condensation

Valentin A. ZAGREBNOV
Département de Mathématiques - Université d'Aix-Marseille

Resumé

- Bose-Einstein Condensation (BEC) of the Free Bose-Gas and **Generalised-BEC** *à la van den Berg-Lewis-Pulé.*
- *One-particle* Integrated Density of States (**IDS**). Examples: BEC in **Magnetic fields**, BEC in "weak" (**scaled**) potentials, BEC in the Interacting Bose-gas with a **spectral gap** in **(I)DS**.
- **Homogeneous Random Potentials:** self-averaging, one-particle **IDS**, *Lifshitz tails* and existence of the **Generalised-BEC**.

• **N.B.** (Random) **external potentials** may reduce the critical dimensionality for the condensation in the Perfect (*not Free*!) Bose-Gas (PBG) to $d_c = 1$!

• **Localization** of the Bose-condensation: the Kac-Luttinger *conjecture* and **Generalized** BEC in extended states.

• What **Random Boson Point Fields (Permanental Processes)** could say more about the condensation in **"Weak Harmonic Traps"** or scaled external potentials ?

• Impact of particle-particle **interaction** (*non-perfect* boson gases). Transformation of the **Generalized** BEC by particle interactions.

• The **Van-der-Waals** interacting bosons in a "weak" external potential.

• Bose-condensation with the **Second Critical Point**.

0. Introduction and Motivation

(a) Bose-Einstein Condensation of a Free Gas: (Einstein-Uhlenbeck-F.London: 1925-38), **Generalised BEC** (Casimir (1968), van den Berg-Lewis-Pulé (1978).

(b) Condensation of Perfect Gases: **Non-Translation-Invariant Condensation in a Restricted Geometry** (Pulé-Verbeure-Z, Martin-Z: 2002), **Condensation in Traps** (Lieb-Solovej-Seilinger-Yngvason: 1999) and (Beau-Tamura-Z: 2012), **in Magnetic and Electrical fields** (Briet-Cornean-Z: 2004, Pulé-Verbeure-Z: 2003-07), **in Random Potentials** (Bru-Dorlas-Lenoble-Pastur-Jaeck-Pulé-Z: 1999-2010), **for Attractive Boundary Conditions** (van den Berg-Lewis-Pulé 1976 and Vandevenne-Verbeure-Z 1999), **Scaled (Weak) Potentials** (van den Berg-Lewis-Pulé-Jaeck-Z 1976-2010)

(c) Dynamical (or **non-conventional**) **Bose Condensation** due to **Interaction** (van den Berg-Dorlas-Lewis-Pulé: 1990 (HYL model) and Bru-Z (**WIBG**): 1998-2008)

(d) Mean-Field and Full Diagonal interacting Bose-gas, Fluctuations and Large Deviations: (Davies-Lewis-Pulé-Buffet-Dorlas-de Smedt-Fannes-Verbeure-Spohn-Lieb-Lebowitz Bru-Z...), **van-der-Waals limit:**(Buffet-de Smedt-Pulé (+ one particle spectral gap) 1983-2009) and (de Smedt-Z (+ weak scaled potential): 1987...)

(e) Interacting Bose-gas with Spectral Gap: (Buffet-de Smedt-Pulé:1983) and (Lauwers-Verbeure-Z: 2003)

(f) Condensation of Non-Perfect Gases = Interacting (+ Non-Free): **There are only few results like Condensation in Traps in the Gross-Pitaevskii limit + some cases of truncated interactions...** (Yngvason-Seiringer-Z 2012).

0*. **Motivation:** revised for the Analysis Seminar

1. In fact the "solution" of the Einstein-Uhlenbeck controversy (1925-38) by F.London in 1938 is an application:

• of an elementary real analysis: uniform / non-uniform convergence (sup-norm) in the **thermodynamic limit** (a ballot on the van der Waals Centenary Conference, 1937);

• of a "scaling limit" (physicists terminology).

2. **EXAMPLE:** Take the sequence $f_n(x) = x^n$ for $x \in [0,1], n \in \mathbb{N}$. Then $\lim_{n\to\infty} f_n(x) = 0$ point-wise on $[0,1)$ and

• there is a uniform convergence: $\lim_{n\to\infty} \sup_{[0,\delta]} f_n(x) = 0$ for $\delta < 1$ but not on $[0,1)$: $\lim_{n\to\infty} \sup_{[0,1)} f_n(x) = 1$

• For any $\alpha \in [0,1)$ there is a sequence $\{x_n(\alpha) \uparrow 1\}_{n\geq 1}$ such that $\lim_{n\to\infty} f_n(x_n(\alpha)) = \alpha$ ("scaling limit").

Ch.1 BEC and GENERALISED CONDENSATION

I Generalized BEC of the Free Bose-Gas

1.1 Free Bose-Gas in $\mathbb{R}^3$

• **Conventional (one-mode) BEC of the free boson gas:** take *cubic box* $\Lambda = L \times L \times L$, $|\Lambda| = V$ with *periodic boundary (p.b.) conditions* for single-particle Hamiltonian $t_\Lambda := (-\Delta/2)_{\Lambda,p.b.}$.

• **Generalized BEC:** take a *parallelepiped* $\Lambda = L_1 \times L_2 \times L_3$ of the *same* volume with sides of length $L_j = V^{\alpha_j}$, $j = 1, 2, 3$, such that $\alpha_1 \geq \alpha_2 \geq \alpha_3 > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$, with the *periodic boundary conditions* for single-particle Hamiltonian $t_\Lambda := (-\Delta/2)_{\Lambda,p.b.}$ on the boundary of Λ (**Casimir boxes** (1968), van den Berg-Lewis-Pulé (1978)).

• *Dual set* Λ^* of momenta w.r.s. to the p.b.:

$$\Lambda^* := \{k_j := 2\pi n_j / V^{\alpha_j} : n_j \in \mathbb{Z}\}_{j=1}^{d=3}\ , \ \text{spec}(t_\Lambda) = \{\varepsilon_k := \sum_{j=1}^{d} k_j^2/2\}_{k \in \Lambda^*}$$

• **Grand-Canonical (β, μ) Free Bose-Gas without QM:**

(a) *Independent* random variables $k \mapsto N_k \in \mathbb{N} \cup \{0\}$, $k \in \Lambda^*$, in the probability space $\Omega := \times_{k \in \Lambda^*} \Omega_k$.

(b) For *bosons* the one-mode random *occupation* numbers are: $N_k \geq 0$ (for *fermions*: $N_k = 0, 1$).

(c) *Probabilities* (**N.B.** for *bosons*: $\mu < 0$, since $\varepsilon_k \geq 0$) :

$$\text{Pr}_{\beta,\mu}(N_k) := e^{-\beta(\varepsilon_k - \mu)N_k} / \Xi_k(\beta, \mu) \ , \ k \in \Lambda^*.$$

(d) *Expectations* for $k \in \Lambda^*$:

$$\mathbb{E}_{\beta,\mu}(N_k) = \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1}.$$

(e) Expectation value of the *total* density of bosons in Λ:

$$\rho_\Lambda(\beta, \mu) := \frac{1}{V} \sum_{k \in \Lambda^*} \mathbb{E}_{\beta,\mu}(N_k).$$

- **Proposition 1.1** Generalized BEC $\neq$ Conventional BEC.
- *Dual set* Λ^* of momenta w.r.s. to the p.b.:

$$\Lambda^* := \{k_j := \frac{2\pi}{V^{\alpha_j}} n_j : n_j \in \mathbb{Z}\}_{j=1}^{d=3} \quad \text{and} \quad \varepsilon_k := \sum_{j=1}^{d} k_j^2/2$$

- **Cube**: $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$, $V = L^3$. If $\mu < 0$ and $\Lambda \nearrow \mathbb{R}^3$:

$$\begin{aligned}
\rho &= \lim_{\Lambda} \rho_\Lambda(\beta, \mu) := \lim_{\Lambda} \frac{1}{V}\{\frac{1}{e^{-\beta\mu} - 1} + \sum_{k \in \{\Lambda^* \setminus \{0\}\}} \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1}\} \\
&= \lim_{L \to \infty} \frac{1}{L^3} \sum_{n_j \in \mathbb{Z} \setminus \{0\}} \left\{ e^{\beta(\sum_{j=1}^{d} (2\pi n_j V^{-1/3})^2/2 - \mu)} - 1 \right\}^{-1} \\
&= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d^3k \left\{ e^{\beta(k^2/2 - \mu)} - 1 \right\}^{-1} =: \mathfrak{I}(\beta, \mu).
\end{aligned}$$

- For $d > 2$ the *free* Bose-gas *critical density*

$$\rho_c(\beta) := \lim_{\mu \nearrow 0} \mathfrak{I}(\beta, \mu) \ ,$$

is finite.

- Then **if** $\rho > \rho_c(\beta) \Rightarrow$ **BEC** at $k = 0$ (ground-state):

$$\rho_0(\beta) := \rho - \rho_c(\beta) \ .$$

1.2 Saturation Mechanism (*conventional* **BEC** *condensation*):

Let $\mu_\Lambda(\beta,\rho)$ be solution of the equation

$$\rho = \rho_\Lambda(\beta,\mu) \quad \Leftrightarrow \quad \rho \equiv \rho_\Lambda(\beta,\mu_\Lambda(\beta,\rho)).$$

- $\lim_\Lambda \ \mu_\Lambda(\beta,\rho<\rho_c(\beta)) = \mu_\Lambda(\beta,\rho) < 0$ or
- $\lim_\Lambda \ \mu_\Lambda(\beta,\rho\geq\rho_c(\beta)) = 0$, and

$$\rho_0(\beta) = \rho - \rho_c(\beta) = \lim_\Lambda \frac{1}{V} \ \left\{e^{-\beta\,\mu_\Lambda(\beta,\rho\geq\rho_c(\beta))} - 1\right\}^{-1} \Rightarrow$$

$$\mu_\Lambda(\beta,\rho\geq\rho_c(\beta)) = -\frac{1}{V}\,\frac{1}{\beta(\rho-\rho_c(\beta))} + o(1/V)\ .$$

- Since $\varepsilon_k = \sum_{j=1}^{d}(2\pi n_j/V^{1/3})^2/2$, the BEC is in **k=0**-mode:

$$\lim_\Lambda \frac{1}{V}\left\{e^{\beta(\varepsilon_{k\neq 0}-\mu_\Lambda(\beta,\rho))} - 1\right\}^{-1} = 0\ ,$$

- This is a well-known *conventional* (**type I**) BEC.

1.3 Saturation Mechanism (*generalised condensation*):

• **The Casimir Box:** Let $\alpha_1 = 1/2$, i.e. $\alpha_2 + \alpha_3 = 1/2$.

Since $\varepsilon_{k_1,0,0} = (2\pi n_1/V^{1/2})^2/2 \sim 1/V$, then again the solution of

$$\rho = \rho_\Lambda(\beta,\mu) \quad \Leftrightarrow \quad \rho \equiv \rho_\Lambda(\beta, \mu_\Lambda(\beta,\rho)).$$

has the asymptotics $\mu_\Lambda(\beta, \rho \geq \rho_c(\beta)) = -A/V + o(1/V)$, $A \geq 0$, although the number of modes producing condensate is **infinite**:

$$\lim_\Lambda \left\{ \frac{1}{V} \frac{1}{e^{-\beta\mu_\Lambda(\beta,\rho)} - 1} + \frac{1}{V} \sum_{k \in \{\Lambda^* : n_1 \neq 0, n_2 = n_3 = 0\}} \frac{1}{e^{\beta(\varepsilon_k - \mu_\Lambda(\beta,\rho))} - 1} \right\}$$
$$= \rho - \rho_c(\beta) > 0.$$

$$\lim_\Lambda \frac{1}{V} \left\{ e^{\beta(\varepsilon_{k\neq 0} - \mu_\Lambda(\beta,\rho))} - 1 \right\}^{-1} \neq 0, \ \text{for } \varepsilon_{k\neq 0} = \varepsilon_{k_1,0,0} \sim \mu_\Lambda(\beta,\rho) \ ,$$

$$\lim_\Lambda \frac{1}{V} \left\{ e^{\beta(\varepsilon_{k\neq 0} - \mu_\Lambda(\beta,\rho))} - 1 \right\}^{-1} = 0, \varepsilon_{0,k_{2,3}\neq 0} \sim (2\pi n_j/V^{\alpha_j})^2/2 > \mu_\Lambda(\beta,\rho).$$

- Generalised BEC **type II** [van den Berg-Lewis-Pulé (1978)]:

$$\rho - \rho_c(\beta) = \lim_{L\to\infty} \frac{1}{V} \sum_{n_1\in\mathbb{Z}} \left\{ e^{\beta((2\pi n_1/V^{1/2})^2/2-\mu_\Lambda(\beta,\rho))} - 1 \right\}^{-1}$$
$$= \sum_{n_1\in\mathbb{Z}} \frac{1}{(2\pi n_1)^2/2 + A} \,.$$

Here $A \geq 0$ is a *unique root* of the above equation.

- **N.B.** For $\alpha_1 = 1/2$ the BEC is still mode by mode **microscopic**, but **infinitely** *fragmented =quasi-condensate.* Experiments with *rotating* condensate (2000) and *chaotic* phases (2008).
- **The van den Berg-Lewis-Pulé Box:** $\alpha_1 > 1/2$.
- **Proposition 1.2** No macroscopic occupation of any level:

$$\lim_{\Lambda} \frac{1}{V} \left\{ e^{\beta(\varepsilon_k-\mu_\Lambda(\beta,\rho))} - 1 \right\}^{-1} = 0.$$

• Generalised BEC **type III** [van den Berg-Lewis-Pulé (1978)]: $\alpha_1 > 1/2$ i.e. $\alpha_2 + \alpha_3 < 1/2$.

• Since $\varepsilon_{k_1,0,0} = (2\pi n_1/V^{\alpha_1})^2/2 \ \sim\ 1/V^{2\alpha_1}$, $2\alpha_1 > 1$, then the solution $\mu_\Lambda(\beta,\rho)$ has **a new** asymptotics: $\mu_\Lambda(\beta,\rho \geq \rho_c(\beta)) = -B/V^\delta + o(1/V^\delta)$, with $B \geq 0$.

• To this end we first must consider the particle density due to summation in k_1-modes:

$$\frac{1}{V}\sum_{k\in\{\Lambda^*:(n_1,0,0)\}} \frac{1}{e^{\beta(\varepsilon_k-\mu_\Lambda(\beta,\rho))}-1} =$$

$$\frac{1}{V}\sum_{k\in\{\Lambda^*:(n_1,0,0)\}} \sum_{s=1}^{\infty} e^{-s\beta(\varepsilon_k-\mu_\Lambda(\beta,\rho))} =$$

$$\frac{1}{V}\sum_{s=1}^{\infty} e^{s\beta\mu_\Lambda(\beta,\rho)} \sum_{n_1=0,\pm1,\pm2,\ldots} e^{-s\beta((2\pi)^2 n_1^2/2V^{2\alpha_1})}.$$

• **N.B.** We are not care very much about $\alpha_2 + \alpha_3 < 1/2$ and about summation over the modes k_2, k_3, since for $n_{2,3} \neq 0$

$$\lim_{\Lambda} \frac{1}{V} \sum_{k \in \{\Lambda^* : (n_1, n_2 \neq 0, n_3 \neq 0)\}} \frac{1}{e^{\beta(\varepsilon_k - \mu_\Lambda(\beta,\rho))} - 1} = \rho_c(\beta), \;\; \rho > \rho_c(\beta).$$

the *Darboux-Riemann* integral-sum converges to $\rho_c(\beta)$.

• For k_1 summation we apply the *Jacobi identity*, with parameter $\lambda = s\beta 2\pi V^{-2\alpha_1}$:

$$\sum_{n_1=0,\pm1,\pm2,\ldots} e^{-\pi\lambda n_1^2} \equiv \frac{1}{\sqrt{\lambda}} \sum_{\xi=0,\pm1,\pm2,\ldots} e^{-(\pi\xi^2/\lambda)} \Rightarrow$$

$$\sum_{n_1=0,\pm1,\pm2,\ldots} e^{-s\beta((2\pi)^2 n_1^2/2V^{2\alpha_1})} = \frac{V^{\alpha_1}}{\sqrt{s\beta 2\pi}} \sum_{\xi=0,\pm1,\pm2,\ldots} e^{-(\pi\xi^2 V^{2\alpha_1}/s\beta 2\pi)} \Rightarrow$$

therefore, only the $\xi = 0$ term survives in the limit $V \to \infty$!

• Thus for the generalized BEC density of the **type III** one obtains:

$$\rho - \rho_c(\beta) =$$

$$\lim_{\Lambda}\left\{(2\pi\beta)^{-1/2}\left\{\frac{V^{\alpha_1-1}}{V^{\delta/2}}\cdot V^{\delta}\right\}\frac{1}{V^{\delta}}\left\{\sum_{s=1}^{\infty} e^{-\beta B\left(s/V^{\delta}\right)}\left(\frac{s}{V^{\delta}}\right)^{-1/2}\right\}\right\}.$$

• This limit is *nontrivial* only for $\delta = 2\,(1-\alpha_1) < 1$:

$$0 < \rho - \rho_c(\beta) = (2\pi\beta)^{-1/2}\int_0^{\infty} d\xi\; e^{-\beta B\xi}\;\xi^{-1/2}\;.$$

• The parameter $B = B\,(\beta,\rho) > 0$ is the *unique* root of the equation:

$$\rho - \rho_c(\beta) = \frac{1}{\sqrt{2\beta^2 B(\beta,\rho)}}\;.$$

• **Generalised BEC of type III**: one-mode particle occupations:

$$\lim_{\Lambda} \frac{1}{V} \langle N_k \rangle_{T_\Lambda} (\beta, \mu_\Lambda (\beta, \rho > \rho_c(\beta))) = 0 \text{ for } \mathbf{all}\ k \in \{\Lambda^*\} \ .$$

• For the *"renormalized"* k_1-modes occupation "density" one obtains:

$$\lim_{\Lambda} \frac{1}{V^{2(1-\alpha_1)}} \langle N_k \rangle_{T_\Lambda} (\beta, \mu_\Lambda (\beta, \rho > \rho_c(\beta))) = 2\beta \left(\rho - \rho_c(\beta)\right)^2 ,$$

where $k \in \{\Lambda^* : (n_1, 0, 0)\}$ and $2(1 - \alpha_1) = \delta < 1$.

• **Definition 1.3** (generalised BEC)

$$\rho - \rho_c(\beta) := \lim_{\eta \to +0} \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*, \|k\| \le \eta\}} \left\{ e^{\beta(\varepsilon_k - \mu_\Lambda(\beta,\rho))} - 1 \right\}^{-1} \ .$$

• **Saturation** ρ_m-**PROBLEM**: [*van den Berg-Lewis-Pulé*] Is it possible that: $\rho_c \le \rho_m \le \infty$ such that **type III (or II)** $\rightarrow$ **type I**, for $\rho \ge \rho_m$? Yes! [Ch.2 BEC with the Second Critical Point].

1.4 Interaction Mechanism

EXAMPLE of creation of generalised condensation III by particle interaction

• Hamiltonian with repulsive interaction (*forward scattering*) and the grand partition function:

$$H_\Lambda^I = \sum_{k\in\Lambda^*} \varepsilon_k a_k^* a_k + \frac{1}{2V}\sum_{k\in\Lambda^*} v(0) a_k^* a_k^* a_k a_k, \quad v(q=0) > 0,$$

$$\Xi_\Lambda^I(\beta,\mu) = Tr_{\mathcal{F}_B} e^{-\beta\left(H_\Lambda^I - \mu N_\Lambda\right)} = \prod_{k\in\Lambda^*}\sum_{n_k=0}^{\infty} e^{-\beta\left[(\varepsilon_k-\mu)n_k + \frac{v(0)}{2V}(n_k^2 - n_k)\right]} .$$

$$p_\Lambda\left[H_\Lambda^I\right] = \frac{1}{\beta V}\ln \Xi_\Lambda^I(\beta,\mu) .$$

- Pressures estimates: $H^I_\Lambda := T_\Lambda + U^{v(0)}_\Lambda \geq T_\Lambda$,

$$p_\Lambda[T_\Lambda] \geq p_\Lambda\left[H^I_\Lambda\right] \geq$$

$$\frac{1}{\beta V}\sum_{k\in\Lambda^*} \ln \sum_{n_k=0}^{[\ln V]} e^{-\beta\left[(\varepsilon_k-\mu)n_k+v(0)(n_k^2-n_k)/2V\right]} \geq$$

$$\frac{1}{\beta V}\sum_{k\in\Lambda^*} \ln \left\{ e^{-\beta v(0)[\ln V]^2/2V}\ \frac{1-e^{-\beta(\varepsilon_k-\mu)([\ln V]-1)}}{1-e^{-\beta(\varepsilon_k-\mu)}} \right\}\Big|_{V\to\infty} =$$

$$p_\Lambda[T_\Lambda]$$

- **Theorem 1.4** $\lim_\Lambda p_\Lambda [T_\Lambda] = \lim_\Lambda p_\Lambda \left[H^I_\Lambda\right]$.

- By the Bogoliubov convexity inequality one obtains:

$$p_\Lambda [T_\Lambda] - p_\Lambda \left[H^I_\Lambda\right] \geq \frac{v(0)}{2} \left\{ \frac{1}{V^2} \sum_{k \in \Lambda^*} \left(\left\langle N_k^2 \right\rangle_{H^I_\Lambda} - \langle N_k \rangle_{H^I_\Lambda} \right) \right\}.$$

- Since for the Gibbs state $\langle - \rangle_{H^I_\Lambda}$ one has

$$\left| \langle A^* B \rangle_{H^I_\Lambda} \right|^2 \leq \langle A^* A \rangle_{H^I_\Lambda} \langle B B^* \rangle_{H^I_\Lambda} \Rightarrow (\langle N_k \rangle_{H^I_\Lambda} / V)^2 \leq \left\langle N_k^2 \right\rangle_{H^I_\Lambda} / V^2.$$

Theorem 1.4 and the Bogoliubov inequality imply :

$$\lim_\Lambda \frac{1}{V} \langle N_k \rangle_{H^I_\Lambda} = 0 \quad k \in \{\Lambda^*\} \ .$$

- Does BEC exist in the model H^I_Λ ?

YES: By Theorem 1.4 and by the Griffiths lemma one has

$$\rho_{c,I}(\beta) = \rho_c(\beta) < \infty \ ,$$

because

$$\rho_I(\beta, \mu < 0) := \lim_\Lambda \partial_\mu p_\Lambda \left[H^I_\Lambda \right] = \lim_\Lambda \partial_\mu p_\Lambda \left[T_\Lambda \right] = \rho(\beta, \mu < 0).$$

II Free Bose-Gas

2.1 One-Particle Integrated Density of States

- Let $\Lambda_L \subset \mathbb{R}^d$, with a smooth boundary $\partial\Lambda_L$ and $|\Lambda_L| = V_L$.
- $\mathcal{H}_L := L^2(\Lambda_L)$, and (free) one-particle Hamiltonian $t_{\Lambda_L} := (-\Delta/2)_{\Lambda_L,D} = t^*_{\Lambda_L}$, with e.g. *D=Dirichlet boundary* conditions.
- t_{Λ_L} has a discrete spectrum $\sigma(t_{\Lambda_L}) = \left\{E_{k,L}\right\}_{k\geq 1}$:

$$t_{\Lambda_L}\psi_{k,L} = E_{k,L}\psi_{k,L}, \;\; 0 < E_{1,L} < E_{2,L} \leq E_{3,L} \leq \dots$$

of finite multiplicity, and $\exp(-\beta t_{\Lambda_L}) \in$ Tr-class($\mathcal{H}_L$) for $\beta > 0$.

Definition 2.1 The finite-volume *integrated density of states* (**IDS**) of t_{Λ_L} is the specific (by a *unit* volume) eigenvalue counting function (*taking multiplicity*)

$$\mathcal{N}_{\Lambda_L}(E) := \max\left\{k : E_{k,L} < E\right\}/|\Lambda_L| \ .$$

Proposition 2.2 There exists a *limiting* integrated density of states: $\mathcal{N}^{(0)}(E) = w - \lim_{L\to\infty}\mathcal{N}_{\Lambda_L}(E)$, where (Weyl):

$$\mathcal{N}^{(0)}(E) = C_d E^{d/2}.$$

2.2 BEC of the Free Bose-Gas

• **Definition 2.3** The grand-canonical **non**-interacting bosons **without** *external potential* are called the (β, μ)-**free** Bose-gas.

• **Proposition 2.4** By the *Bose-statistics* and by **Definition 2.1** of the *finite-volume* **IDS**, the *mean value* of the *total* particle-density $\rho_{\Lambda_L}(\beta, \mu)$ in the volume Λ_L is:

$$\rho_{\Lambda_L}(\beta, \mu) = \frac{1}{V} \sum_{\psi_{k,L}} \frac{1}{e^{\beta(E_{k,L}-\mu)} - 1} = \int_0^\infty \frac{\mathcal{N}_{\Lambda_L}(dE)}{e^{\beta(E-\mu)} - 1} \ , \ \mu < 0 \ .$$

• By **Proposition 2.2**, the limiting density $\rho(\beta, \mu)$ exists for *negative* chemical potentials $\mu \in (-\infty, 0)$:

$$\rho(\beta, \mu) = \int_0^\infty \frac{\mathcal{N}^{(0)}(dE)}{e^{\beta(E-\mu)} - 1} = -\int_0^\infty dE \; \mathcal{N}^{(0)}(E) \partial_E \left\{ \frac{1}{e^{\beta(E-\mu)} - 1} \right\} \; .$$

• The **critical** density $\rho_c(\beta) := \rho(\beta, -0) < \infty$ is **finite** for $d > d_c = 2$, since

$$\mathcal{N}^{(0)}(dE) \sim E^{d/2-1} \; dE \; ,$$

(the *Weyl formula*).

We resume the above observations as the main statement about the generalised BEC (à la van den Berg-Lewis-Pulé) for the case of the free boson gas:

• **Proposition 2.6** Let $\rho_c(\beta) < \infty$ and $\mu_{\Lambda_L}(\beta, \rho)$ be unique root of equation $\rho = \rho_L(\beta, \mu)$. For $\rho \geq \rho_c(\beta)$, $\lim_{L\to\infty} \mu_{\Lambda_L}(\beta, \rho) = 0$ and the BEC density $\rho_0(\beta, \rho) := \rho - \rho_c(\beta) > 0$ is

$$\rho_0(\beta, \rho) = -\lim_{\epsilon \downarrow 0} \lim_{L\to\infty} \int_0^{\epsilon} dE \; \mathcal{N}_{\Lambda_L}(E) \; \partial_E \left\{ \frac{1}{e^{\beta(E-\mu_{\Lambda_L}(\beta,\rho))} - 1} \right\}$$

• **N.B.** If $\rho_c(\beta) = \infty$, this statement has no sense, **but** the value of critical density $\rho_c(\beta)$ may be **changed**, if the non-interacting gas is placed in an **external potential**: since the value of $\rho_c(\beta)$ is a function of the critical dimensionality d_c and the latter is a functional of the **One-Particle Density of States**: $\mathcal{N}^{(0)}(dE)$.

2.3 Why the Bose Condensation is a Subtle Matter ?

- Let $\Lambda_{L,\mathbf{D}} = \times_{j=1}^{3}[-L/2, L/2]$ be a **cube**. Then the density:

$$\rho_0(\beta, \rho > \rho_c(\beta)) = \lim_{L\to\infty} \frac{1}{L^3} \left\{ e^{\beta(E_{k,L}-\mu_L(\beta,\rho))} - 1 \right\}^{-1} \delta_{\mathbf{1},k}$$

$$= \lim_{L\to\infty} (\rho - \rho_c(\beta))\delta_{\mathbf{1},k}, \; E_{\mathbf{1}=(1,1,1),L} = \frac{1}{2}\{3(\pi/L)^2\} \to E_{gr} = 0.$$

is the **ground-state BEC** (*type* I), $E_{\mathbf{1},L} - \mu_L(\beta, \rho)) \sim L^{-3}$.

$E_{gr} = 0 \bullet - - - - - - - E_* \bullet - - - - - - - - - - - - - - - \longrightarrow$ **E**

- **If** $\rho_c(\beta) = \int_0^\infty \mathcal{N}^{(0)}(dE)\{e^{\beta E} - 1\}^{-1} = \infty$ $\Leftrightarrow$ **high** *density of states* $\mathcal{N}^{(0)}(dE)$ at $E = 0$ (e.g. $E^{d/2-1}dE$ for $d \leq 2$) $\Leftrightarrow$ a "*leaking*" of the *type* I condensate into **excited** states $\Rightarrow$

• **Conclusion:** To *preserve* the BEC one has to *suppress* density of states in the *vicinity* of the **ground-state** ($E_{gr} = 0$), e.g., a **spectral gap**: $\mathcal{N}^{(0)}(E) = \theta(E - E_{gr})$ for $E < E_*$, where $E_{gr} < E_*$ [Buffet,Pulé,Lauwers,Verbeure,Z].

III Perfect Bose-Gas in Magnetic Field

3.1 Hamiltonian

- Let open $\Lambda_{L=1} \subset \mathbb{R}^{d=3}$ with $|\Lambda_{L=1}| = 1$ and piecewise continuously differentiable boundary $\partial\Lambda_{L=1}$ contain the origin $\{x = 0\}$. Put $\Lambda_L := \{x \in \mathbb{R}^3 : L^{-1}x \in \Lambda_{L=1}\}$, $L > 0$.
- Take a magnetic *vector-potential* in the form: $A(x) = \omega\ A_0(x)$, $\omega \geq 0$. For two types of *gauges*: symmetric (*transverse*): $A_t(x) = 1/2(-x_2, x_1, 0)$, or *Landau*: $A_l(x) = (0, x_1, 0)$, this generates a constant unit magnetic field $\mathbf{B} = \text{rot}\ A$, *parallel* to the third direction OX_3.
- The one-particle Hamiltonian with *Dirichlet* boundary conditions (D) on $\partial\Lambda_L$ is defined in $L^2(\Lambda_L)$ by

$$h_{\Lambda_L}(\omega) := (-i\nabla - A)^2 + V_{\Lambda_L} \equiv t_{\Lambda_L}(\omega) + V_{\Lambda_L}\ ,$$

where V_{Λ_L} is an *eventual* external "*electric*" potential. Then $h_{\Lambda_L}(\omega)$ has purely discrete spectrum.

3.2 No-Go Theorem for BEC in a Constant Magnetic Field

• Let a continuous external potential $V(x) = v(x_1)$ (v be $\mathbb{Z}$-periodic) and we use the *Landau gauge*: $A_l(x) = (0, x_1, 0) \in \mathbb{R}^3$. Then the *bulk* Hamiltonian acting in $L^2(\mathbb{R}^3)$, where $\omega \geq 0$, is:

$$h_\infty(\omega) = (-i\nabla - \omega A_l)^2 + v = -\partial_{x_1}^2 + v(x_1) + (-i\partial_{x_2} - \omega x_1)^2 - \partial_{x_3}^2 \ .$$

• **Proposition 3.1** Let $E_0(\omega) := \inf \sigma(h_\infty(\omega))$. Then: for $E \searrow E_0(\omega)$ one gets:

$$\mathcal{N}_{\infty,\omega}(E) = B_{\omega,d} \cdot (E - E_0(\omega))^{(d-2)/2} + o((E - E_0(\omega))^{d/2-1}).$$

Hence, for $d = 3$ and any $\omega > 0$ the critical density

$$\rho_c(\beta) = - \lim_{\mu \nearrow E_0(\omega)} \int_{E_0(\omega)}^{\infty} dE \ \mathcal{N}_{\infty,\omega}(E) \partial_E \left\{ \frac{1}{e^{\beta(E-\mu)} - 1} \right\} = \infty,$$

is infinite, i.e. the BEC is destroyed by a *constant* magnetic field (1958). **N.B.** *Weyl*: $\mathcal{N}^{(0)}(E) \sim E^{d/2}$.

• **Remark 3.2** Operator $h_\infty(\omega)$ is unitary equivalent to the sum of ω-harmonic oscillator (Landau levels) and one-dimensional v-Schrödinger operator in the *third direction*. If $v = 0$, then

$$\mathcal{N}_{\infty,\omega}(E) = \omega(E-\omega)^{1/2}/2\pi^2$$

between the first two Landau levels: $E \in (\omega, 3\omega)$, i.e.

$$d = 3 \text{ and } \omega > 0 \Leftrightarrow d = 1 \text{ and } \omega = 0$$

• **Proposition 3.3 [BCZ (2004)]** Assume that $\omega = 2\pi$. Then there exists an external "*electric*" potential of the form:

$$V_\epsilon(x) = \epsilon \cdot [v_1(x_1) + v_2(x_2)] + v_3(x_3),$$

where $\epsilon > 0$ and small, each of the functions $\{v_j\}_{j=1}^3$ is a smooth $\mathbb{Z}$-periodic potential, and *neither* one of v_1 and v_2 is constant, that critical density is bounded.

IV Bose-Condensation in Random Potentials

4.1 Random Schrödinger Operator (RSO)

• **Random Repulsive Impurities:** $u(x) \geq 0$, $x \in \mathbb{R}^d$, continuous function with a *compact* support is a local single-impurity potential. The *Poisson* Random Potential (*PRP*):

$$v^{\omega}(x) := \int_{\mathbb{R}^d} \mu_{\tau}^{\omega}(dy) u(x-y) = \sum_j u(x - y_j^{\omega}) \geq 0, \ \omega \in \Omega.$$

where impurity positions $\left\{y_j^{\omega}\right\} \subset \mathbb{R}^d$ are the atoms of the random Poisson measure:

$$\mathbb{P}\left(\{\omega \in \Omega : \mu_{\tau}^{\omega}(\Lambda) = n\}\right) = \frac{(\tau\,|\Lambda|)^n}{n!} e^{-\tau|\Lambda|},$$

$n \in \mathbb{N} \cup \{0\}$, $\Lambda \subset \mathbb{R}^d$, $\mathbb{E}\left(\mu_{\tau}^{\omega}(\Lambda)\right) = \tau\,|\Lambda|$, the parameter τ is concentration of impurities.

• **Proposition 4.1** The spectrum $\sigma(h^\omega)$ of $\{h^\omega := t + v^\omega\}_{\omega\in\Omega}$ is almost-surely (a.s.) non-random and coincides with $[0, +\infty)$.

• RECALL of SOME GENERAL RANDOM POTENTIAL PROPERTIES

In the framework of general setting this model corresponds to the following one-dimensional ($d = 1$) single-particle *random* Schrödinger operator in the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$:

• Consider a random (measurable) potential $v^{(\cdot)}(\cdot) : \Omega \times \mathbb{R} \to \mathbb{R}, (\omega, x) \mapsto v^{\omega}(x)$, which is a random field on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the properties:

(a) v^{ω} is homogeneous and ergodic with respect to the group $\{\tau_x\}_{x\in\mathbb{R}}$ of probability-preserving translations on $(\Omega, \mathcal{F}, \mathbb{P})$;

(b) v^{ω} is non-negative and $\inf_{x\in\mathbb{R}^d} \{v^{\omega}(x)\} = 0$.

By $\mathbb{E}\{\cdot\} := \int_{\Omega} \mathbb{P}(d\omega)\{\cdot\}$ we denote the expectation with respect to the probability measure in $(\Omega, \mathcal{F}, \mathbb{P})$.

• Then the *random* Schrödinger operator corresponding to the potential v^ω is a family of random operators $\{h^\omega\}_{\omega\in\Omega}$:

$$h^\omega := t + v^\omega, \tag{1}$$

where $t := (-\Delta/2)$ is the *free* one-particle Hamiltonian, i.e., a unique self-adjoint extension of the operator: $-\Delta/2$, with domain in $L^2(\mathbb{R})$.

• Notice that assumptions (a) and (b) guarantee that there exists a subset $\Omega_0 \subset \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that operator (1) is *essentially* self-adjoint on domain $\mathcal{C}_0^\infty(\mathbb{R})$ for every $\omega \in \Omega_0$.

4.2 Self-Averaging of IDS and Lifshitz Tail [L.Pastur]

- The restriction $h_L^\omega := (-\Delta/2 + v^\omega)_{\Lambda_L, \mathcal{D}}$ has a (*random*) finite-volume **IDS**:

$$\mathcal{N}_L^\omega(E) := \frac{1}{|\Lambda_L|} \max\left\{ j : \phi_j^\omega, E_j^\omega(L) < E \right\},\ \omega \in \Omega.$$

- **Proposition 4.2** There exists a non-random distribution $\mathcal{N}(E)$ (*measure* $\mathcal{N}(dE)$) such that (*a.s.*)

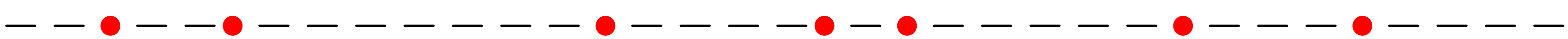
$$w - \lim_{L\to\infty} \mathcal{N}_L^\omega(E) = \mathcal{N}(E)\ ,\ \sigma(h^\omega) = \text{supp}\,\mathcal{N}(dE)\ ,$$

with (*non-random*) lower edge $E_0 = 0$.

- **Proposition 4.3** (*Lifshitz tail*) The asymptotics of $\mathcal{N}(E)$ as $E \downarrow 0$:

$$\mathcal{N}(E)|_{E\downarrow 0} \sim \exp\left\{-\tau\,(c_d/E)^{d/2}\right\}, \text{ with } \tau \geq 0 \text{ and } c_d > 0.$$

- The *self-averaging* of the limiting **IDS** is true for the Poisson *point-impurities*: $u(x) = a\,\delta(x)$, $a > 0$, on the line $\mathbb{R}^1$:



4.3 BEC of the Perfect Bose-Gas in the Poisson Random Potential

- The *random* finite-volume bosons particle density:

$$\rho_L^\omega(\beta,\mu) = \int_0^\infty \mathcal{N}_L^\omega(dE)\frac{1}{e^{\beta(E-\mu)}-1}$$

for $\beta > 0$, $\mu < 0$ and any realization $\omega \in \Omega$.

- **Proposition 4.4** By Proposition 3.2 the limit:

$$a.s. - \lim_{L\to\infty} \rho_L^\omega(\beta,\mu) = \int_0^\infty \frac{\mathcal{N}(dE)}{e^{\beta(E-\mu)}-1} \equiv \rho(\beta,\mu),$$

uniformly in μ on compacts in $(-\infty, 0)$.

- **Corollary 4.5** The *Lifshitz tail* implies that $\rho_c(\beta) := \rho(\beta,-0) < \infty$ for $d > 0$, so there is a condensation of the Perfect Bose-Gas for *low dimensions* $d = 1,2$.

• **Proposition 4.6 [Lenoble-Pastur-Zagrebnov (2004)]** Let $\rho \geq \rho_c(\beta)$ and $\mu_L^\omega(\beta,\rho)$ be a unique root of equation $\rho = \rho_L^\omega(\beta,\mu)$ for $\omega \in \Omega$. Then $a.s. - \lim_{L\to\infty} \mu_L^\omega(\beta,\rho) = 0$, and:

$$\lim_{\epsilon \downarrow 0} \left\{ a.s. - \lim_{L\to\infty} \int_0^\epsilon \mathcal{N}_L^\omega(dE) \frac{1}{e^{\beta(E-\mu_L^\omega(\beta,\rho))} - 1} \right\}$$

$$(a.s.) = \rho - \rho_c(\beta) = \rho_0(\beta,\rho) \geq 0 \ .$$

• A.s. *non-random* $\rho_0(\beta,\rho)$ is a generalized condensation density à la van den Berg-Lewis-Pulé.

• **Kac-Luttinger Conjecture (1973-74).** *For PBG in the one-dimensional random Poisson potential of (point) impurities the Bose-condensate is of the type I and it is localized* **around** *the one "largest box" on the line* $\mathbb{R}^1$, corresponding to the support ($\sim \ln(L)$) of the **ground state** ϕ_1.

• **Proposition 4.7 [Z, Lenoble-Z (2007)]** The Kac-Luttinger conjecture is true for one-dimensional hard ($a = +\infty$) Poisson random point impurities: the BEC for the PBG is of the *type I* and it is *localized* in one "largest box".

$- - \bullet - - \bullet - - - - - - - \sim \ln(L) - - - - - - - - - \bullet - - - - - - \bullet -$

V Can we save the Bogoliubov Theory (BT) for External Random Potentials ?

5.1 Random Eigenfunctions/Kinetic-Energy Eigenfunctions

• Recall that for the random Schrödinger operator in $\Lambda \subset \mathbb{R}^d$:

$$h_{\Lambda}^{\omega}\, \phi_j^{\omega} = (t_{\Lambda} + v^{\omega})_{\Lambda}\, \phi_j^{\omega} = E_j^{\omega}\, \phi_j^{\omega} \ , \text{ for a.a. } \omega \in \Omega \ .$$

• Let $N_{\Lambda}(\phi_j^{\omega})$ be particle-number operator in the eigenstate ϕ_j^{ω}.

$$N_{\Lambda} := \sum_{j \geq 1} N_{\Lambda}(\phi_j^{\omega}) := \sum_{j \geq 1} a^*(\phi_j^{\omega}) a(\phi_j^{\omega})$$

is the *total* number operator in the boson Fock space $\mathfrak{F}_B(L^2(\Lambda))$, where $a(\phi_j^{\omega}) := \int_{\Lambda} dx\, \overline{\phi_j^{\omega}}(x)\, a(x)$, and $\{\phi_j^{\omega}\}_{j \geq 1}$ is a basis in $L^2(\Lambda)$.

• Let $t_{\Lambda}\, \psi_k = \varepsilon_k\, \psi_k$ be the kinetic-energy operator eigenfunctions and eigenvalues $\varepsilon_k = \hbar^2\, k^2/2m$. One of the **key hypothesis** of the **BT** is the ground-state (or zero-mode $\psi_{k=0}$) condensation.

5.2 The First Main Theorem

• **Theorem 5.1**[Jaeck-Pulé-Zagrebnov (2009)]
Let $H^\omega_\Lambda := T_\Lambda + V^\omega_\Lambda + U_\Lambda$ be many-body Hamiltonians of interacting bosons in random external potential (trap) V^ω_Λ. If particle *interaction* U_Λ commutes with any of the operators $N_\Lambda(\phi^\omega_j)$, then

$$\lim_{\delta\downarrow 0}\lim_{\Lambda}\sum_{j:E^\omega_j\leqslant\delta}\frac{1}{V}\langle(N_\Lambda(\phi^\omega_j))\rangle_{H^\omega_\Lambda}>0 \Leftrightarrow$$

$$\Leftrightarrow \lim_{\gamma\downarrow 0}\liminf_{\Lambda}\sum_{k:\varepsilon_k\leqslant\gamma}\frac{1}{V}\langle N_\Lambda(\psi_k)\rangle_{H^\omega_\Lambda}>0\ ,$$

and: $\lim_{\gamma\downarrow 0}\lim_\Lambda\sum_{k:\varepsilon_k>\gamma}\langle N_\Lambda(\psi_k)\rangle_{H^\omega_\Lambda}/V = 0$. Here $\langle - \rangle_{H^\omega_\Lambda}$ is the quantum Gibbs expectation with Hamiltonians H^ω_Λ.

• **Corollary 5.2** The localised (*random*) generalized boson condensation occurs **if and only if** there is a generalized condensation in the extended (*kinetic-energy*) eigenstates.

5.2 The Second Main Theorem

- Let for any $A \subset \mathbb{R}_+$ the particle occupation measures m_Λ and $\tilde{m}_\Lambda$ are defined by:

$$m_\Lambda(A) := \frac{1}{V} \sum_{j:E_i \in A} \langle N_\Lambda(\phi_i^\omega) \rangle_{H_\Lambda^\omega} \ , \ \tilde{m}_\Lambda(A) := \frac{1}{V} \sum_{k:\varepsilon_k \in A} \langle N_\Lambda(\psi_k) \rangle_{H_\Lambda^\omega} \ .$$

- **Theorem 5.3**[Jaeck-Pulé-Zagrebnov (2009)]

$$m(dE) = \begin{cases} (\overline{\rho} - \rho_c)\delta_0(dE) + (e^{\beta E} - 1)^{-1} \mathcal{N}(dE) & \text{if } \overline{\rho} \geqslant \rho_c \ , \\ (e^{\beta(E - \mu_\infty)} - 1)^{-1} \mathcal{N}(dE) & \text{if } \overline{\rho} < \rho_c \ , \end{cases}$$

$$\tilde{m}(d\varepsilon) = \begin{cases} (\overline{\rho} - \rho_c)\delta_0(d\varepsilon) + F(\varepsilon)d\varepsilon & \text{if } \overline{\rho} \geqslant \rho_c, \\ F(\varepsilon)d\varepsilon & \text{if } \overline{\rho} < \rho_c. \end{cases}$$

with explicitly defined density $F(\varepsilon)$.

- **Corollary 5.4**

Densities of **generalised** *random* and *kinetic-energy* states condensates **coincide !**

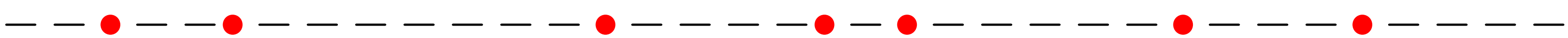
5.3 Example: BEC in One-Dimensional Random Potential. Poisson Point-Impurities

- For $d = 1$ *Poisson point-impurities*, $a > 0$:

$$v^{\omega}(x) := \int_{\mathbb{R}^1} \mu_{\tau}^{\omega}(dy) a\, \delta(x-y) = \sum_j a\, \delta(x - y_j^{\omega})$$

Proposition 5.5 Let $a = +\infty$. Then $\sigma(h^{\omega})$ is a.s. nonrandom, dense *pure-point* spectrum $\overline{\sigma_{p.p.}(h^{\omega})} = [0, +\infty)$, with IDS

$$\mathcal{N}(E) = \tau \frac{e^{-\pi\tau/\sqrt{2E}}}{1 - e^{-\pi\tau/\sqrt{2E}}} \sim \tau e^{-\pi\tau/\sqrt{2E}}\,,\ E \downarrow \mathbf{0}$$



- **Spectrum:**

$$(a.s.) - \sigma(h^\omega) = \bigcup_j \left\{\pi^2 s^2/2(L_j^\omega)^2\right\}_{s=1}^{\infty}$$

- Intervals $L_j^\omega = y_j^\omega - y_{j-1}^\omega$ are *i.i.d.r.v.* :

$$dP_{\tau,j_1,\ldots,j_k}(L_{j_1},\ldots,L_{j_k}) = \tau^k \prod_{s=1}^{k} e^{-\tau L_{j_s}} dL_{j_s}$$

- **Eigenfunctions:**

One-particle **localized** quantum states $\{\phi_j\}_{j\geq 1}$, a basis in $L^2(\Lambda)$.

5.4 BEC in One-Dimensional Nonrandom Potential: Point-Impurities(*hierarchical* model [LZ (2007)])

- Let $[0, L] = \bigcup_{j=1}^{n} I_j$, $I_j = [y_{j-1}, y_j]$, $y_0 = 0, y_n = L$ and $v(x) := \sum_{j=0}^{n} a\, \delta(x - y_j)\, , a = +\infty$
- Let $h_0(I_j) := (-\Delta/2)_{I_j,\mathcal{D}}$. The model: $h_L := (-\Delta/2) \dot{+} v(x) = \bigoplus_{j=1}^{n-1} h_0(I_j)$, $L_j = \left| I_j \right|$

$$\sigma(h_L) = \bigcup_{j=1}^{n-1} \left\{ E_s(L_j) \equiv \pi^2 s^2 / 2(L_j)^2 \right\}_{s=1}^{\infty} , (p.p.)$$

- Let $L_{j=2,3,\ldots} = (L - L_1)/(n-1) \equiv \tilde{L}$ and $L_1 = f(L) < L$: $\lim_{L\to\infty} f(L)/L = 0$.

- Finite-volume *total* particle density :

$$\rho_L(\beta,\mu) = \frac{1}{L}\sum_{s=1}^{\infty}\left\{e^{\beta(E_s(L_1)-\mu)} - 1\right\}^{-1} +$$
$$\frac{n-1}{L}\sum_{s=1}^{\infty}\left\{e^{\beta(E_s(\tilde{L})-\mu)} - 1\right\}^{-1} \ , \ \ \mu \leq 0$$

- For $\tau = \lim_{n,L\to\infty} n/L = \lim_{n,L\to\infty} \tilde{L}^{-1}$ the *critical* density $\rho_c(\beta) := \lim_{\mu\nearrow 0}\lim_{n,L\to\infty}\rho_L(\beta,\mu)$.

$$\rho_c(\beta) = \tau\sum_{s=1}^{\infty}\left\{e^{\beta E_s(\tau^{-1})} - 1\right\}^{-1} < \infty$$

5.5 BEC in One-Dimensional Nonrandom Potential (I)

- Let $\rho_L(\beta, \mu_L(\beta,\rho)) = \rho < \rho_c(\beta)$. Then $\lim_{L\to\infty} \mu_L(\beta,\rho) = \tilde{\mu}(\beta,\rho) < 0$ and

$$\rho = \tau \sum_{s=1}^{\infty} \left\{ e^{\beta[E_s(\tau^{-1}) - \tilde{\mu}(\beta,\rho)]} - 1 \right\}^{-1}$$

- $L_1 = L^{1/2-\epsilon}$: Let $\rho_L(\beta, \mu_L(\beta,\rho)) = \rho \geq \rho_c(\beta)$ and $L_1 = f(L) = L^{1/2-\epsilon}$, $\epsilon > 0$. Then

$$\mu_L(\beta,\rho) = \pi^2/2L_1^2 - (\beta L(\rho - \rho_c(\beta)))^{-1} + O(L^{-2})$$

- BEC density $\rho_0(\beta,\rho) = \rho - \rho_c(\beta)$:

$$\rho_0(\beta,\rho) = \lim_{n,L\to\infty} \frac{1}{L} \sum_{s=1}^{\infty} \left\{ e^{\beta(E_s(L_1) - \mu_L(\beta,\rho))} - 1 \right\}^{-1}$$

$$= \lim_{n,L\to\infty} \frac{1}{L} \left\{ e^{\beta(E_1(L_1) - \mu_L(\beta,\rho))} - 1 \right\}^{-1}$$

This is the *ground-state* (**type I**) BEC, *localized* in the *largest* box $L_1 \to \infty$.

- **Type II BEC:** $L_1 = L^{1/2}$

Then

$$\mu_L(\beta,\rho) = -A(\beta,\rho)/L + O(L^{-2})$$

and BEC is *fragmented* among **infinitely** many levels in *one largest* box.

5.6 BEC in One-Dimensional Nonrandom Potential (II)

• This is the **type II** *generalized* BEC in the *largest* box, with **infinitely** many (single-particle) levels **macroscopically** occupied:

$$\rho - \rho_c(\beta) = \lim_{n,L\to\infty} \frac{1}{L} \sum_{s=1}^{\infty} \frac{1}{e^{\beta(E_s(L_1)-\mu_L(\beta,\rho))} - 1}$$

$$= \sum_{s=1}^{\infty} \left\{\beta(\pi^2 s^2/2 + A(\beta,\rho))\right\}^{-1} \ , \ \ A(\beta,\rho) > 0$$

• $L_1 = L^{1/2+\epsilon}$: One gets the **type III** *generalized* BEC in the *largest* box: **none** of single-particle levels is **macroscopically** occupied.

- Chemical potential:

$$\mu_L(\beta,\rho) = -B(\beta,\rho)/L^{1-2\epsilon} + O(L^{-1})$$

and

$$\rho - \rho_c(\beta) = \lim_{n,L\to\infty} \frac{1}{L} \sum_{s=1}^{\infty} \frac{1}{e^{\beta(E_s(L_1)-\mu_L(\beta,\rho))} - 1}$$

$$= \frac{1}{\sqrt{2\pi\beta}} \int_0^{\infty} dt e^{-\beta t B(\beta,\rho)} t^{-1/2} \ , \ \ B(\beta,\rho) > 0$$

5.7 Nonrandom/Random Potential (III)

•*Spatially* fragmented *type III* BEC in the *hierarchical* model splitted between (*infinitely*) many *different* intervals:

$$L_j = \frac{\ln(\lambda L)}{\lambda}, 1 \leq j \leq [\ln(k+1)] =: M_k,$$

$$L_{j>M_k} = \tilde{L}_k := \frac{L - L_1 M_k}{k - M_k}$$

$$\rho_L(\beta,\mu) = \frac{1}{L}\sum_{j=1}^{M_k}\sum_{s=1}^{\infty}\frac{1}{e^{\beta(c^2s^2/L_j^2-\mu)}-1} +$$

$$\frac{k-M_k}{L}\sum_{s=1}^{\infty}\frac{1}{e^{\beta(c^2s^2/\tilde{L}_k^2-\mu)}-1}$$

$\lim_{L\to\infty}\tilde{L}_k = \lim_{L\to\infty} L/(k-M_k) = 1/\lambda$, condensate $\rho - \rho_c(\beta) = \rho_0(\beta,\rho) > 0$ is *equally* splitted between *infinitely* many intervals.

VI Off-Diagonal-Long-Range-Order (ODLRO)

6.1 BEC of the Free Bose-Gas: ODLRO

- *PBG one-body reduced density matrix*:

$$\rho_L(\beta, \mu; x, y) = \sum_{k \geq 1} \frac{1}{e^{\beta(E_k(L)-\mu)} - 1} \overline{\psi_{k,L}(x)} \psi_{k,L}(y)$$

Its *diagonal* part is the *local* particle number density.

Proposition 6.1 For the free Bose-gas ($L \to \infty$)

$\rho(\beta, \mu(\beta, \rho); x, y) =$

$$\begin{cases} \sum_{s=1}^{\infty} (2\pi\beta s)^{-d/2} e^{s\beta\mu(\beta,\rho) - \|x-y\|^2/2\beta s}, \rho < \rho_c(\beta) \\ \rho_0(\beta, \rho) \left|\psi_{k,L=1}(0)\right|^2 + \sum_{s=1}^{\infty} \frac{e^{-\|x-y\|^2/2\beta s}}{(2\pi\beta s)^{d/2}}, \rho \geq \rho_c(\beta) \end{cases}$$

Here $\rho_0(\beta, \rho)$ is the condensate density and $\psi_{k=1,L=1}(0)$ is the ground state eigenfunction in domain $\Lambda_{L=1}$ evaluated at the point of dilation $x = 0$.

- **Definition 6.2** The *Off-Diagonal Long-Range Order*:

$$ODLRO(\beta, \rho) := \lim_{\|x-y\| \to \infty} \rho(\beta, \mu(\beta, \rho); x, y)$$

6.2 One-Body Reduced Density Matrix for Random Potentials

- *Space averaged* reduced density matrix

$$\tilde{\rho}_L^\omega(\beta,\mu;x,y) := \frac{1}{|\Lambda_L|}\int_{\Lambda_L} da\rho_L^\omega(\beta,\mu;x+a,y+a)$$

- For non-negative measurable ergodic random potentials, any $\mu < 0$ and any fixed $x, y \in \mathbb{R}^d$ one gets *self-averaging* of the *reduced density matrix*:

$$a.s. - \lim_{L\to\infty} \tilde{\rho}_L^\omega(\beta,\mu;x,y) = \tilde{\rho}(\beta,\mu;x-y)$$

Proposition 6.3 Then

$$\rho(\beta, \mu - \tau\tilde{u}; x - y) \le \tilde{\rho}(\beta, \mu; x - y) \le \rho(\beta, \mu; x - y),$$

where $\tilde{u} := \int_{\mathbb{R}^1} dx u(x)$.

Proposition 6.4 Let $\mu < 0$. For one-dimensional Poisson potential with $\operatorname{supp} u(x) = [-\delta/2, \delta/2]$

$$\tilde{\rho}(\beta, \mu; x - y) \le \rho(\beta, \mu; x - y) e^{-\tau\tilde{\gamma}(|x-y|-\delta)},$$

where $\tilde{\gamma} := 1 - e^{-\tilde{u}}$.

Corollary 6.5 If impurity concentration $\tau \downarrow 0$:

$$\lim_{\tau \downarrow 0} \tilde{\rho}(\beta, \mu; x - y) = \rho(\beta, \mu; x - y)$$

VII Kac-Luttinger Conjecture [KL (1973-74]

• *In the case of the one-dimensional random Poisson potential of point impurities the BEC for the PBG is of the type I and it is localized in one "largest box".*

7.1 Statistics of Poisson Intervals:

• Consistent *marginals* in the (thermodynamic) limit $\lambda = \lim_{L\to\infty} n/L$ have the form:

$$d\sigma_{\lambda,k}(L_{j_1}, \ldots, L_{j_k}) = \lambda^k \prod_{s=1}^{k} e^{-\lambda L_{js}} dL_{j_s} \ .$$

• Expectation value of the intervals length:

$$\mathbb{E}_{\sigma_\lambda}(L^{\omega}_{j_s}) = \lambda \int_0^{\infty} dL\, L\, e^{-\lambda L} = \lambda^{-1} \ .$$

- For ordered intervals:

$$\left\{ L^\omega_{j_1} \geq L^\omega_{j_2} \geq \ldots \geq L^\omega_{j_k} : \sum_{s=1}^{k} L^\omega_{j_s} = L^\omega \simeq k/\lambda (LLN) \right\} \ ,$$

the joint distribution for k intervals on $\mathbb{R}$ is

$$d\sigma^{>}_{\lambda,k}(L_{j_1}, \ldots, L_{j_k}) := k!\ \theta(L_{j_1} - L_{j_2}) \ldots \theta(L_{j_{k-1}} - L_{j_k})\ d\sigma_{\lambda,k}(L_{j_1}, \ldots, L_{j_k}) \ .$$

- **Proposition**[LZ(2007)] Let $d\sigma^{>}_{L,\lambda,k}$ be joint distribution of k intervals $\sum_{s=1}^{k} L^\omega_{j_s} = L$. Then

$$\lim_{L\to\infty} \frac{\mathbb{E}_{\sigma_{L,\lambda}}(L^\omega_{j_1})}{\ln(\lambda L)} = \frac{1}{\lambda} \ .$$

- Probabilities of the "energies repulsions" in different boxes:

$$\mathbb{P}\{\omega : L^\omega_{j_1} - L^\omega_{j_2} > \delta\} = e^{-\lambda\delta},\ \delta > 0.$$

7.2 Application of the Borel-Cantelli Lemma

- Energies in the *samples* $\left\{|I_j^\omega(k)| = L_j^\omega(k)\right\}_{j=1}^k$:

$$E_s(L_{j_r}^\omega(k)) = \frac{c^2 s^2}{(L_{j_r}^\omega(k))^2}\ ,\ r = 1, ..., k\ ,\ s = 1, 2, ...\ .$$

- Let the events $(k = 1, 2, \dots)$

$S_k(a > 0, 0 < \gamma < 1) := \{\omega : E_{s=1}(L_{j_2}^\omega(k)) - E_{s=1}(L_{j_1}^\omega(k)) > \frac{a}{k^{1-\gamma}}\}$

- Since $\lim_{k\to\infty} \mathbb{P}\{S_k(a, 0 < \gamma < 1)\} = 1$, one gets *divergence*

$$\lim_{k\to\infty} \sum_{r=1}^{k} \mathbb{P}\{S_k(a, \gamma)\} = \infty.$$

• Then *independence* of the *events* $\{S_k(a,\gamma)\}_{k=1}^{\infty}$ and the well-known *Borel-Cantelli* lemma imply:

$$\mathbb{P}\,\{\overline{\lim_{k\to\infty}}\, S_k(a,\gamma)\} = 1,\ \overline{\lim_{k\to\infty}}\, S_k(a,\gamma) = \bigcap_{k=1}^{\infty}\bigcup_{l=k} S_l(a,\gamma)$$

• Notice that the event:

$$\overline{\lim}\, S_k(a,\gamma) := \bigcap_{k=1}^{\infty}\bigcup_{l=k} S_l(a,\gamma)$$

means that *infinitely* many events $\{S_k(a,\gamma)\}_{k\geq 1}$ take place.
• This means (in turn) that with the probability 1 the BEC is localized in the thermodynamic limit $\mathbb{R}$ in a single "*largest box*", and this condensation is of the *type* I.

VIII Bose Condensation in Scaled ("Weak") Potentials

8.1 Bose-Gas in a Scaled ("Weak") Potential

• **Definition 8.1** Let $v(x) \geq 0$ be continuous function: $v \in C(\mathbb{R}^d)$ such that $v(0) = 0$. We say that $\{V_L\}_L$ is a family of the scaled ("Weak") potentials in the box $\Lambda_L = L \times L \times \ldots \times L \ni \mathcal{O}$, if

$$(v_L \phi)(x) := v(x/L)\phi(x) \ , \ x \in \mathbb{R}^d \ , \ \phi \in \mathcal{H} = L^2(\Lambda_L)$$

• For the *perfect* boson gas the many-body problem in external potential reduces to the one-particle problem:

$$h_{\Lambda_L} \phi^v_{j,L} = \left(h_{0,L} + v_L\right) \phi^v_{j,L} = E^v_{j,L} \phi^v_{j,L} \ ,$$

with some boundary conditions on $\partial\Lambda_L$.

• The finite-volume *integrated density of states* (**IDS**) in the presence of external potential (see Definition 2.1):

$$\mathcal{N}^v_{\Lambda_L}(E) := \text{card} \left\{\phi^v_{j,L} : E^v_{j,L} < E\right\} / |\Lambda_L| \ .$$

• Then the *mean-value* of the perfect Bose-gas *total* particle-density is:

$$\rho_{\Lambda_L}(\beta,\mu) = \frac{1}{|\Lambda_L|}\sum_{\psi^v_{j,L}} \frac{1}{e^{\beta(E^v_{j,L}-\mu)}-1} = \int_0^\infty \frac{\mathcal{N}^v_{\Lambda_L}(dE)}{e^{\beta(E-\mu)}-1} \ , \ \mu < 0 \ .$$

• The criterium of the Bose-condensate is the value of the *critical density*:

$$\rho_{c,v}(\beta) := \sup_{\mu<0} \lim_{L\to\infty} \int_0^\infty \frac{\mathcal{N}^v_{\Lambda_L}(dE)}{e^{\beta(E-\mu)}-1} = \int_0^\infty \frac{\mathcal{N}^v(dE)}{e^{\beta E}-1} \ .$$

• Our next problem is calculation of the limiting **IDS** in external potential v:

$$\mathcal{N}^v(E) := w - \lim_{L\to\infty} \mathcal{N}^v_{\Lambda_L}(E) \ .$$

8.2 Integrated Density of States in External Potential v_L

- (a) Laplace transformation ($t > 0$):

$$\begin{aligned}\Phi_{\Lambda_L}(\tau) := & \int_0^\infty \mathcal{N}^v_{\Lambda_L}(dE)\ e^{-tE} = \frac{1}{|\Lambda_L|} \sum_{\{\phi^v_{j,L}\}} e^{-tE^v_{j,L}} \\ = & \frac{1}{|\Lambda_L|} \mathrm{Tr}_{L^2(\Lambda_L)}\ e^{-t(h_{0,L}+v_L)} \\ = & \frac{1}{|\Lambda_L|} \sum_{k>1} \int_{\Lambda_L} dx\ \overline{u_k}(x)\ (e^{-t(h_{0,L}+v_L)}\ u_k)(x)\ ,\end{aligned}$$

$\{u_k\}_{k>1}$ is any orthonormal basis in the Hilbert space $\mathcal{H} = L^2(\Lambda_L)$.

- (b) **Free propagator** e^{-th_0} , $(e^{-th_0} f)(x) := \int_{\mathbb{R}^d} dy G_0(x,y;t) f(y)$.

$$(e^{-th_0})(x,y) := G_0(x,y;t) = \frac{1}{(4\pi D t)^{d/2}} e^{-\|x-y\|^2/(4Dt)} \ ,$$
$$h_0 = -\Delta/2 \ ,$$

($\|x-y\|$ *Euclidean distance*), is the Green function for the *heat equation*):

$$\partial_\tau G_0 = D \ \Delta_x G_0 \ , G_0(x,y;\tau=0) = \delta_y(x) \ ,$$
$$D = 1/2 \ .$$

• (c) Kernel of the *perturbed* propagator $e^{-\tau(t_L+v_L)}$, $\tau > 0$

The Lie-Trotter (1875-1958) product formula for the perturbed propagator:

$$e^{-t\,(h_{0,L}+v_L)} = \lim_{n\to\infty} \left(e^{-(t/n)h_{0,L}} e^{-(t/n)v_L}\right)^n \; .$$

$$f_t(x) = \lim_{n\to\infty} (e^{-(t/n)h_{0,L}} e^{-(t/n)v_L} \; \ldots \; e^{-(t/n)h_{0,L}} e^{-(t/n)v_L} \; f_0)(x).$$

$$f(x) \mapsto (e^{-(t/n)v_L} f)(x) = e^{-(t/n)v(x/L)} \; f(x) \; , \; f \in L^2(\mathbb{R}^d) \; .$$

Therefore, the Lie-Trotter product formula for the perturbed propagator implies:

$$f_t(x) = \lim_{n\to\infty} \int_{\Lambda_L} dx_{n-1} \; G_{0,L}(x, x_{n-1}; t/n) e^{-tv(x_{n-1}/L)/n}$$
$$\ldots \int_{\Lambda_L} dx_0 \; G_{0,L}(x_1, x_0; t/n) e^{-tv(x_0/L)/n} f_0(x_0) \; . \Rightarrow$$

- (d) **Wiener Path Integral:**

$$f_t(x) = \lim_{n\to\infty} \int_{\Lambda_L} dx_0 \ldots \int_{\Lambda_L} dx_{n-1} \int_{\Omega^{t/n}_{x,x_{n-1}}\cap\Lambda_L} d\mu^{t/n}_{x,x_{n-1}}(\omega) \ldots$$

$$\int_{\Omega^{t/n}_{x_1,x_0}\cap\Lambda_L} d\mu^{t/n}_{x_1,x_0}(\omega) e^{-\frac{t}{n}\sum_{j=0}^{n-1} v(x_j/L)} f_0(x_0) =$$

$$\lim_{n\to\infty} \int_{\Omega^t_x\cap\Lambda_L} d\mu^t_x(\omega)\ e^{-\frac{t}{n}\sum_{j=0}^{n-1} v(\omega(jt/n)/L)} f_0(\omega(0)),$$

$\omega(jt/n) = x_j$.

Here the *one-point* conditional Wiener measure $d\mu^t_x(\omega)$ on the set Ω^t_x verifies the condition:

$$\int_{\Omega^t_{x,y}} d\mu^t_{x,y}(\omega) = \int_{\Omega^t_x} d\mu^t_x(\omega)\delta_y(\omega(t)) \ .$$

• Recall that $d\mu^t_{x,y}(\omega)$ is the two-point *conditional* Wiener measure on the space of the Wiener pathes:

$$\{\omega \in \Omega^t_{x,y} : \ \omega(t) = x, \ \omega(0) = y\} \ , \Omega^t_x = \bigcup_{y \in \mathbb{R}^d} \Omega^t_{x,y} \ .$$

• This conditional Wiener measure on the set $\Omega^t_{x,y}$ verifies:

$$\int_{\Omega^t_{x,y}} d\mu^t_{x,y}(\omega) \equiv G_0(x, y; t) \geq 0 \ .$$

• (e) **Feynman-Kac Formula**

Since $\frac{t}{n}\sum_{j=0}^{n-1} v(\omega(jt/n)/L)$ is the Darboux-Riemenn sum, in limit $n \to \infty$ one gets the Feynman-Kac formulae:

$$f_t(x) = \int_{\Omega_x^t \cap \Lambda_L} d\mu_x^t(\omega)\; e^{-\int_0^t ds\, v(\omega(s)/L)}\; f_0(\omega(0)) = (e^{-t\,(h_{0,L}+v_L)} f_0)(x) \; .$$

$$f_t(x) = \int_{\Lambda_L} dy G_{v_L}(x,y;t) f_0(y) \Leftrightarrow G_{v_L}(x,y;t) = (e^{-t\,(h_{0,L}+v_L)})(x,y) \; .$$

$$(e^{-t\,(h_{0,L}+v_L)})(x,y) = \int_{\Omega_{x,y}^t \cap \Lambda_L} d\mu_{x,y}^t(\omega)\;\; e^{-\int_0^t ds\, v(\omega(s)/L)}$$

$$(e^{-t\,(h_{0,L}+v_L)})(x,y)|_{v_L=0} = \int_{\Omega_{x,y}^t \cap \Lambda_L} d\mu_{x,y}^t(\omega) = G_{0,L}(x,y;t)$$

$$\lim_{L\to\infty} (e^{-t\,h_{0,L}})(x,y) = \int_{\Omega_{x,y}^t} d\mu_{x,y}^t(\omega) = \frac{1}{(4\pi D t)^{d/2}} e^{-\|x-y\|^2/(4Dt)} \; .$$

• (f) **Limiting IDS for the Scaled Potential** v_L

(1)

$$\Phi_{\Lambda_L}(t) = \frac{1}{|\Lambda_L|} \sum_{n>1} \int_{\Lambda_L} dx \ \overline{u_n}(x) \int_{\Lambda_L} dy \ G_{v_L}(x,y;t) \ u_n(y) =$$

$$\frac{1}{|\Lambda_L|} \int_{\Lambda_L} dx \int_{\Lambda_L} dy \ G_{v_L}(x,y;t) \sum_{n>1} \ \overline{u_n}(x) \ u_n(y) =$$

$$\frac{1}{|\Lambda_L|} \int_{\Lambda_L} dx G_{v_L}(x,x;t) = \frac{1}{|\Lambda_L|} \int_{\Lambda_L} dx \int_{\Omega^t_{x,x} \cap \Lambda_L} d\mu^t_{x,x}(\omega) \ e^{-\int_0^t ds \ v(\omega(s)/L)} \ .$$

(2)

$$\{x^\alpha/L = z^\alpha\}_{\alpha=1}^d \quad , \quad dx/|\Lambda_L| = dz \ , \ z \in \Lambda_{L=1} \ .$$

$$\omega(s) = x - \frac{s}{t}(x-y) + \tilde{\omega}(s) \ , \ \tilde{\omega}(s=0) = \tilde{\omega}(s=t) = 0 \ .$$

(3)

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \int_{\Lambda_L} dx \int_{\Omega^t_{x,x}\cap\Lambda_L} d\mu^t_{x,x}(\omega)\ e^{-\int_0^t ds\ v((x+(s/t)(x-y)+\tilde{\omega}(s))/L)|_{x=y}}$$

$$= \lim_{L\to\infty} \int_{\Lambda_{L=1}} dz \int_{\Omega^t_{Lz,Lz}} d\mu^t_{Lz,Lz}(\omega)\ e^{-\int_0^t ds\ v(z+\tilde{\omega}(s)/L)}$$

$$= \int_{\Lambda_{L=1}} dz\ \frac{e^{-t\ v(z)}}{(2\pi t)^{d/2}} = \int_0^\infty \mathcal{N}(dE)\ e^{-tE}\ .$$

(4) By the inverse Laplace transformation one finds for $\mathcal{N}(dE) = n(E)dE$ and $a_d = (\Gamma(d/2))^{-1}$:

$$n(E) := \int_{\Lambda_{L=1}} dz\ \theta(E - v(z))\ (E - v(z))^{d/2-1}\ \frac{a_d}{(2\pi)^{d/2}}\ ,$$

where $a_d/(2\pi)^{d/2} = (d/2)C_d$ and $C_d = \frac{1}{(4\pi)^{d/2}\Gamma((d/2)+1)}$ (Weyl):
$v = 0 \Rightarrow \mathcal{N}_0(dE) = C_d\ d/2\ E^{d/2-1}\ dE$.

8.3 Bose-Condensation in External Potential v_L

• The criterium of the Bose-condensate is the value of the *critical density*:

$$\rho_{c,v}(\beta) := \sup_{\mu<0} \lim_{L\to\infty} \int_0^\infty \frac{\mathcal{N}^v_{\Lambda_L}(dE)}{e^{\beta(E-\mu)}-1} = \int_0^\infty \frac{\mathcal{N}^v(dE)}{e^{\beta E}-1} =$$

$$\int_{\Lambda_{L=1}} dz \int_0^\infty dE\ \theta(E-v(z))\ (E-v(z))^{d/2-1} \frac{(d/2)C_d}{e^{\beta(E-\mu)}-1} =$$

$$\int_0^\infty d\varepsilon\ \varepsilon^{d/2-1} \int_{\Lambda_{L=1}} dz\ \frac{(d/2)C_d}{e^{\beta(\varepsilon+v(z)-\mu)}-1}\ .$$

• **Example 8.2** Let $d = 1$ and $v(x) = |x|$. Then

$$\begin{aligned} \rho_{c,d=1,v}(\beta) &= (1/2)C_1 \int_0^\infty d\varepsilon\ \varepsilon^{-1/2} \int_{\Lambda_{L=1}} dx \ \sum_{s=1}^{\infty} e^{-s\beta\varepsilon}\ e^{-s\beta|x|} \\ &= 2\ C_1 \ \sum_{s=1}^{\infty} \int_0^\infty \frac{1}{\sqrt{s}}\ d\eta\ e^{-\beta\eta^2} \ \int_0^{s/2} \frac{1}{s}\ dw\ e^{-\beta w} \\ &\leq C_1\ \frac{\sqrt{\pi}}{\beta^{3/2}} \ \sum_{s=1}^{\infty} \ \frac{1}{s^{3/2}} \end{aligned}$$

is bounded. Perfect boson gas manifests condensation even for $d = 1$, if it is trapped by a "weak" potential $v(x/L) = |x|/L$.

• Notice that $\rho_{c,d=1,v}(\beta) = \infty$ for a "weak" harmonic potential $v(x) = x^2$!

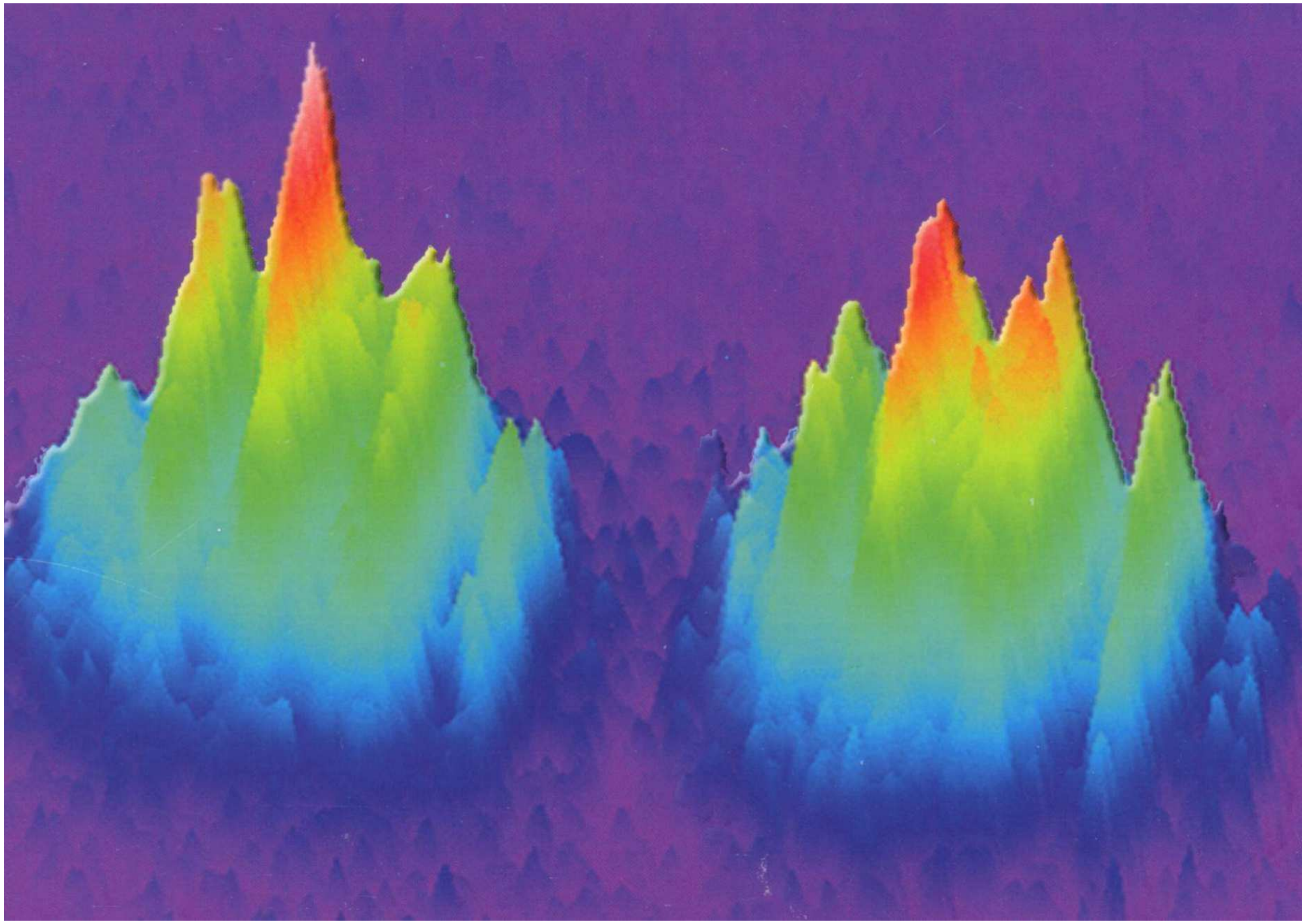
Ch.2 BEC with the SECOND CRITICAL POINT

- 0.Experimental Data.
- 1.Perfect Bose-gas.
- 2.Exponential SLAB and the Second Critical Point.
- 3.Exponential BEAM and CIGAR Traps.
- 4.Temperature Dependence of the Bose-Condensate.
- 5.Anisotropy and Localisation.
- 6.Coherence Length and Anisotropy.

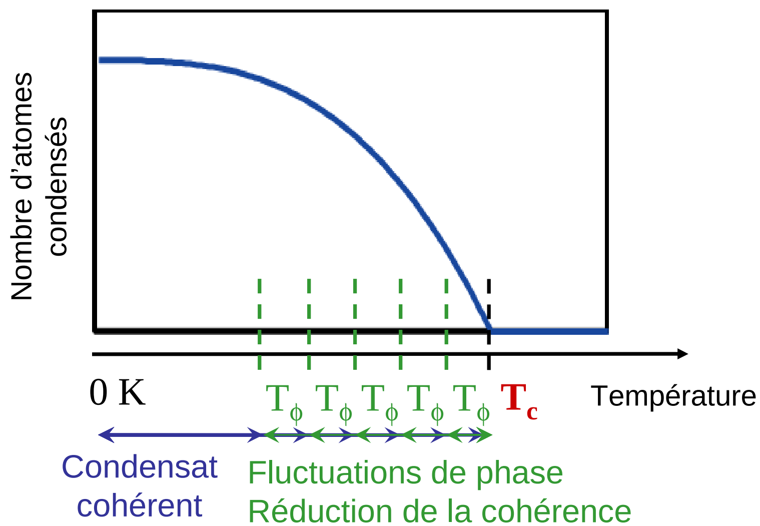
M.Beau, V.A.Z. arXiv:1002.1242, Cond.Mat.Phys.**31**, 23003:1-10 (2010)

W.J.Mullin, A.R.Sakhel J.Low Temp.Phys.**166**,125-150 (2012)

0. Experimental Data

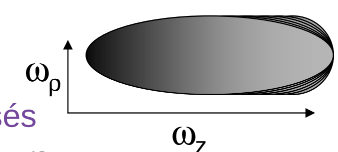


Température de phase T_ϕ (caractérise la cohérence)



$$T_\phi = \frac{15}{32} \frac{(\hbar\omega_z)^2 N}{\mu}$$

T_ϕ petite :
- condensat long
- peu d'atomes condensés



D. Petrov *et al.* [PRL **87**, 050404 (2001)]

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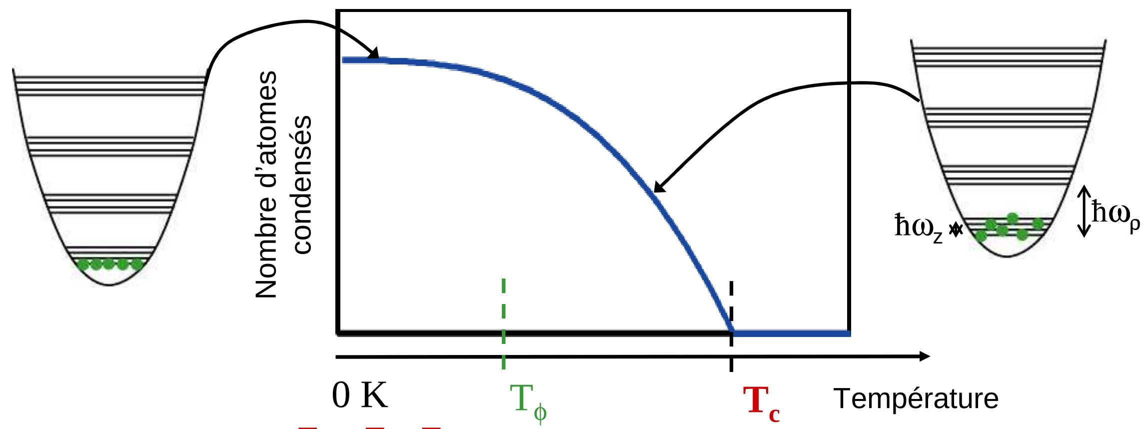
Origine des fluctuations de phase

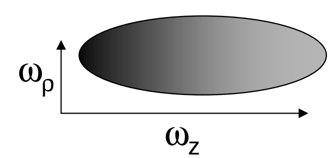


$T_\phi < T < T_c$:

Distribution aléatoire sur plusieurs niveaux d'énergie très proches

⇒ Fluctuations de phase suivant l'axe long du condensat

Amplitude des fluctuations de phase : $\dfrac{T}{T_\varphi}$



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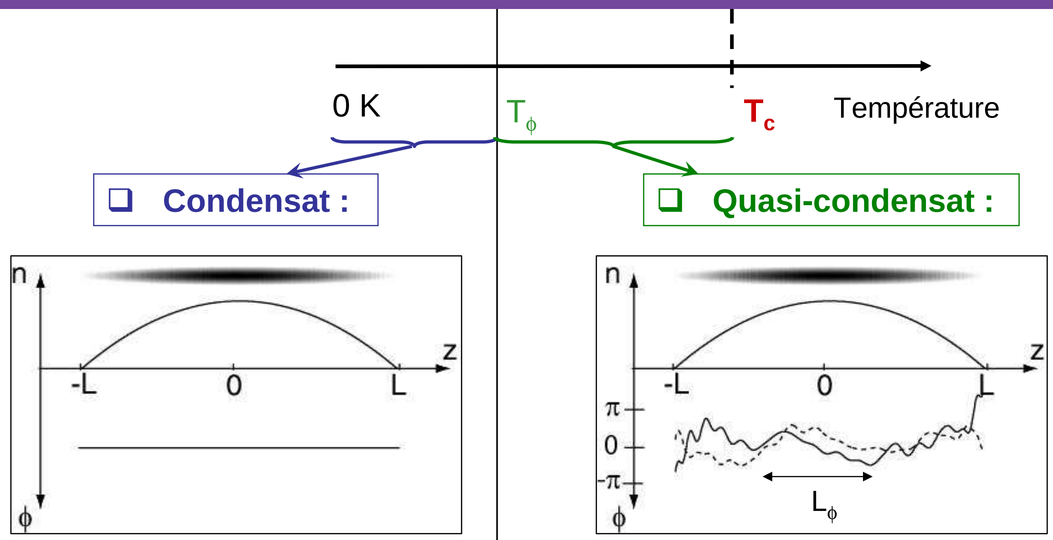
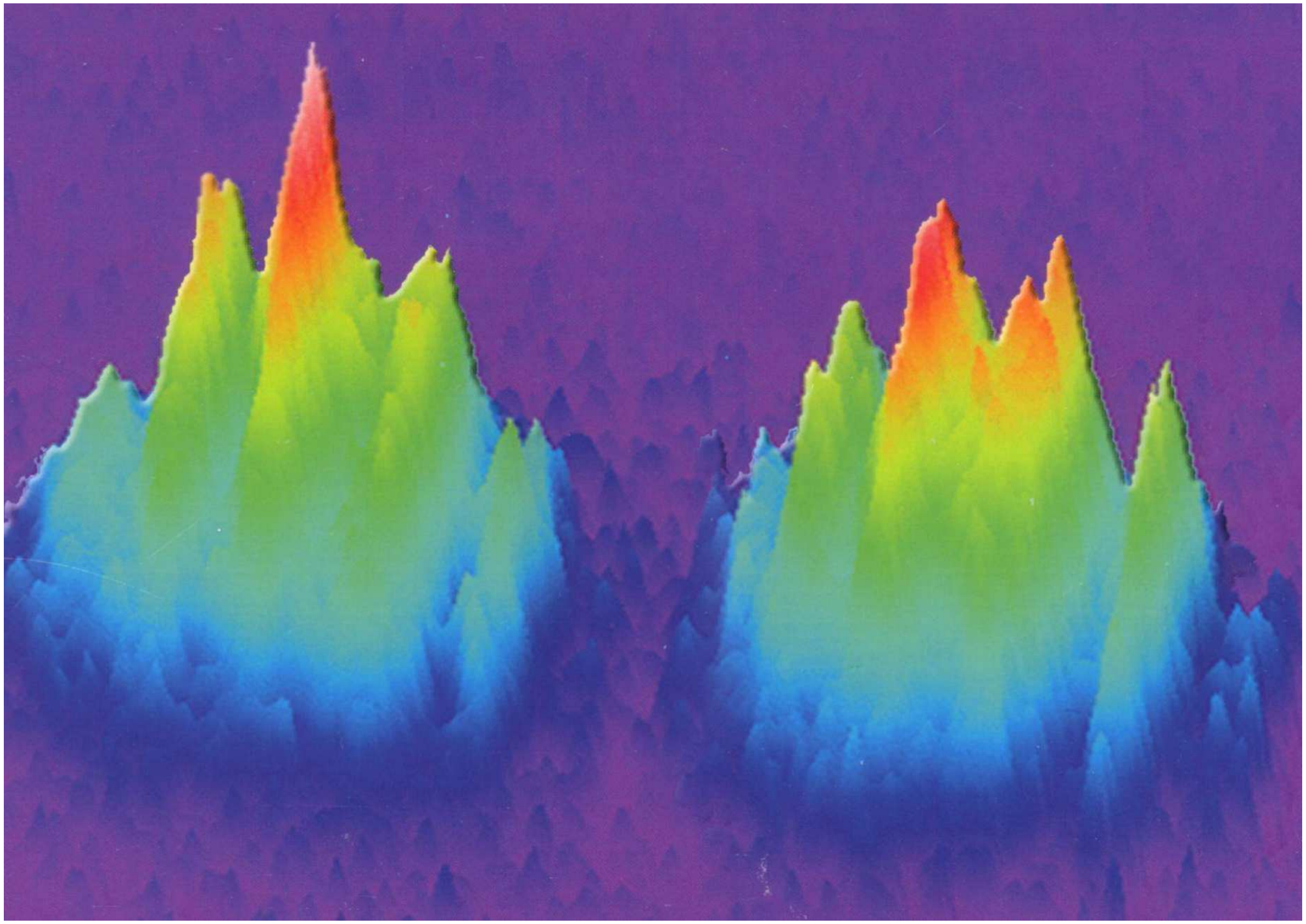
Densité et phase du quasi-condensat



Phase φ est uniforme
$L_c = L$

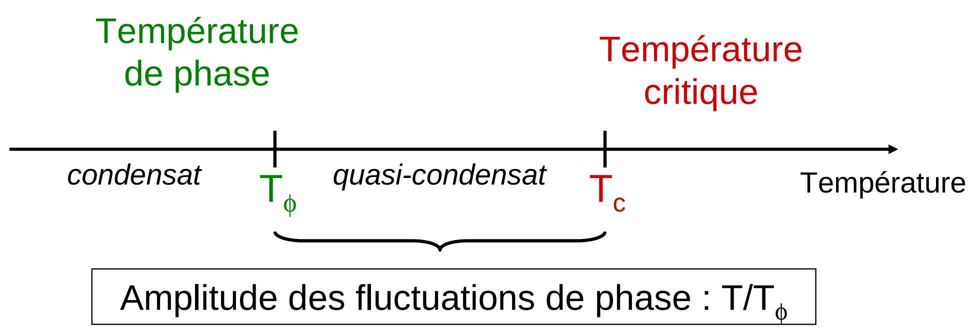
Phase fluctue suivant l'axe long du piège
$L_c < L$

10



Résumé

- Deux températures pour la caractérisation de la condensation :



1. Perfect Bose-gas

- For $\Lambda = L_1 \times L_2 \times L_3 \in \mathbb{R}^3$ and $T_\Lambda^{(N=1)} = \left(-\hbar^2 \Delta/(2m)\right)_D$ the spectrum:

$$\left\{ \varepsilon_s = \frac{\hbar^2}{2m} \sum_{j=1}^{3} (\pi s_j / L_j)^2 \right\}_{s_j \in \mathbb{N}}$$

- Eigenfunctions: $\{\phi_{s,\Lambda}(x) = \prod_{j=1}^{3} \sqrt{2/L_j} \sin(\pi s_j x_j / L_j)\}_{s_j \in \mathbb{N}}$, $s := (s_1, s_2, s_3) \in \mathbb{N}^3$
- In (T, V, μ), $V = L_1 L_2 L_3$ the Gibbs mean occupation number of $\phi_{s,\Lambda}$ is $N_s(\beta, \mu) = (e^{\beta(\varepsilon_s - \mu)} - 1)^{-1}$, $\mu < \inf_s \ \varepsilon_s$.
- Particle density $\rho_\Lambda(\beta, \mu) = \sum_{s \in \mathbb{N}^3} N_s(\beta, \mu)/V =: N_\Lambda(\beta, \mu)/V$
- The **first** critical density: $\rho_c(\beta) := \sup_{\mu \leq 0} \lim_\Lambda \rho_\Lambda(\beta, \mu) = \zeta(3/2)/\lambda_\beta^3$, $\lambda_\beta := \hbar\sqrt{2\pi\beta/m}$, de Broglie thermal length.

2. Exponential SLAB and the Second Critical Point

2.1 Let $\Lambda = Le^{\alpha L} \times Le^{\alpha L} \times L$. Then for $\mu \leq 0$ we have

$$\lim_{L\to\infty} \sum_{(s_1,s_2,s_3\neq 1)} \frac{N_s(\beta,\mu)}{V_L} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{d^3k}{e^{\beta(\hbar^2 k^2/2m-\mu)} - 1} \,.$$

2.2 Let $\mu_L(\beta,\rho) := \varepsilon_{(1,1,1)} - \Delta_L(\beta,\rho)$, where $\Delta_L(\beta,\rho) \geq 0$ is a unique solution of the equation:

$$\rho = \sum_{s=(s_1,s_2,1)} \frac{N_s(\beta,\mu)}{V_L} + \sum_{(s_1,s_2,s_3\neq 1)} \frac{N_s(\beta,\mu)}{V_L}. \qquad (2)$$

2.3 Since: $\lim_{L\to\infty}\sum_{(s_1,s_2,s_3\neq 1)} N_s(\beta,\mu=0)/V_L = \rho_c(\beta)$, for $\rho > \rho_c(\beta)$ the limit of the first sum is

$$\lim_{L\to\infty} \sum_{s=(s_1,s_2,1)} \frac{N_s(\beta,\mu_L(\beta,\rho))}{V_L} =$$

$$\lim_{L\to\infty} \frac{1}{L}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{d^2k}{e^{\beta(\hbar^2k^2/2m+\Delta_L(\beta,\rho))}-1} =$$

$$\lim_{L\to\infty} -\frac{1}{\lambda_\beta^2 L}\ln[\beta\Delta_L(\beta,\rho)] = \rho-\rho_c(\beta).$$

This implies the asymptotics:

$$\Delta_L(\beta,\rho>\rho_c(\beta)) = \frac{1}{\beta}\, e^{-\lambda_\beta^2(\rho-\rho_c(\beta))L} + \dots\ .$$

2.4 Remark 2.1. Since $L_{j=1,2} = Le^{\alpha L}$ and

$$\varepsilon_{(s_1,s_2,1)} - \mu_L(\beta,\rho) = \frac{\hbar^2}{2m}\sum_{j=1}^{2}[(\pi s_j/L_j)^2 - 1] + \Delta_L(\beta,\rho)$$

the representation of the first sum by the **integral** is valid only when $\lambda_\beta^2(\rho - \rho_c(\beta)) \leq 2\alpha$. It is *implied* by **2.3** and the estimate:

$$\frac{\hbar^2}{2m}\ \pi^2 L^{-2} e^{-2\alpha L} < \Delta_L(\beta,\rho) = \beta^{-1}\ e^{-\lambda_\beta^2(\rho-\rho_c(\beta))L} + \ldots\ .$$

2.5 Definition 2.2. The **second** critical density:

$$\rho_m(\beta) := \rho_c(\beta) + 2\alpha/\lambda_\beta^2 > \rho_c(\beta)\ .$$

2.6 Remark 2.3. For $\rho > \rho_m(\beta)$ the convergence $\Delta_L(\beta,\rho) \to 0$ *should* be **faster** than $e^{-2\alpha L}$.

2.7 To keep the difference $\rho - \rho_m(\beta) > 0$ one **must** return back to the finite volume **sum representation** to take into account the input of the ground state occupation density.

Theorem 2.4. The asymptotics of $\Delta_L(\beta, \rho > \rho_m(\beta))$ is

$$\Delta_L(\beta,\rho) = [\beta(\rho - \rho_m(\beta))V_L]^{-1} + \ldots < \pi^2 L^{-2} e^{-2\alpha L}\ \hbar^2/2m\ .$$

2.8 Since $V_L = L^3 e^{2\alpha L}$, the first sum without the ground-state:

$$\lim_{L\to\infty} \sum_{s=(s_1>1, s_2>1, 1)} \frac{N_s(\beta,\mu)}{V_L} = 2\alpha/\lambda_\beta^2 = \rho_m(\beta) - \rho_c(\beta)$$

$$[= \lim_{L\to\infty} \frac{1}{L}\frac{1}{(2\pi)^2}\int_{\|k\|>\pi/Le^{-\alpha L}} \frac{d^2k}{e^{\beta(\hbar^2 k^2/2m + \Delta_L(\beta,\rho>\rho_m))} - 1}].$$

2.9 The ground-state term gives the **macroscopic** occupation:

$$\rho - \rho_m(\beta) = \lim_{L\to\infty} \frac{1}{V_L}\frac{1}{e^{\beta(\varepsilon_{(1,1,1)} - \mu_L(\beta,\rho))} - 1}\ .$$

2.10 Corollary 2.5 Since for $\rho_c(\beta) < \rho < \rho_m(\beta)$

$$\varepsilon_s - \mu_L(\beta,\rho) = \Delta_L(\beta,\rho) + \varepsilon_s - \varepsilon_{(1,1,1)} = \mathcal{O}(\beta^{-1}\ e^{-\lambda_\beta^2(\rho-\rho_c(\beta))L})\ ,$$

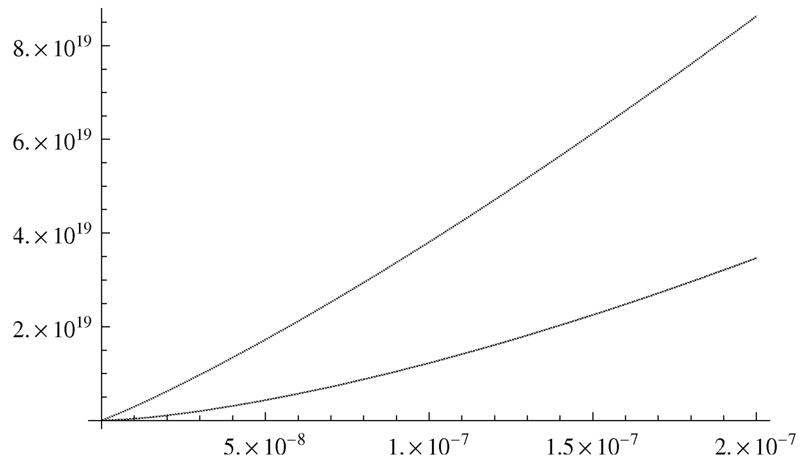
one gets the *type* III van den Berg-Lewis-Pulé generalised condensation (vdBLP-GC): when **none** of the single-particle states are *macroscopically* occupied:

$$\rho_s(\beta,\rho) := \lim_{L\to\infty} \frac{1}{V_L} \frac{1}{e^{\beta(\varepsilon_s-\mu_L(\beta,\rho))} - 1} = 0\ .$$

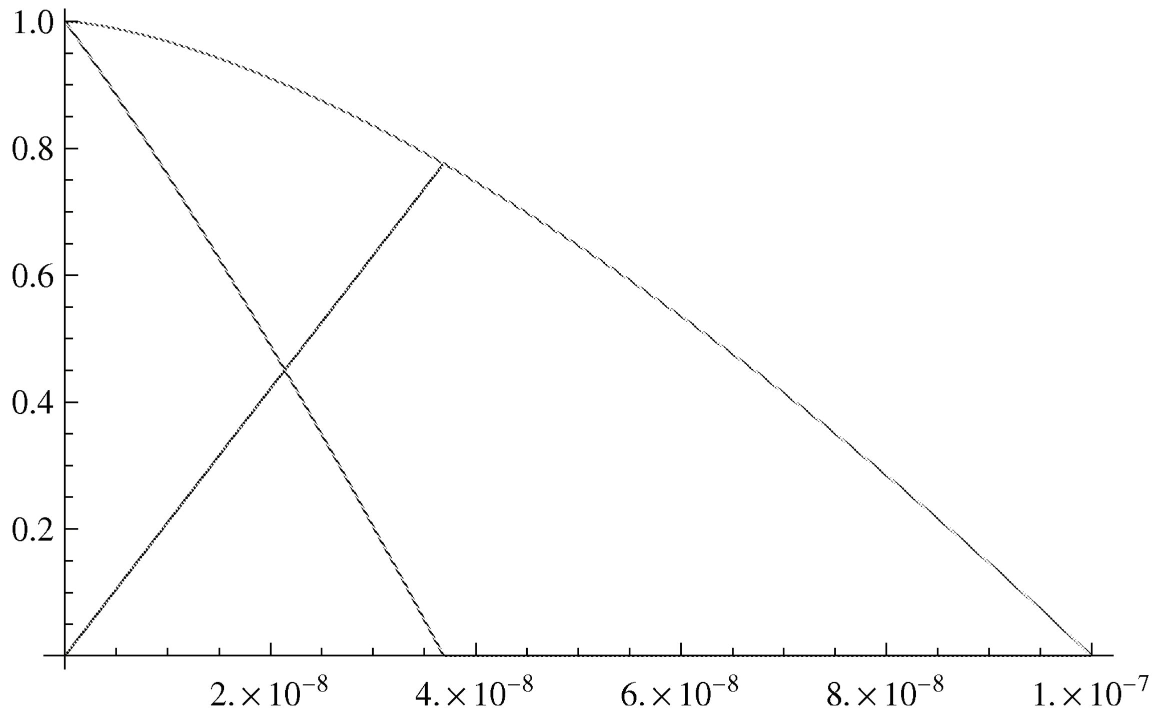
The asymptotics $\Delta_L(\beta,\rho > \rho_m(\beta)) = [\beta(\rho-\rho_m(\beta))V_L]^{-1}$ implies

$$\lim_{L\to\infty} \rho_{s\neq(1,1,1)}(\beta,\rho) := \lim_{L\to\infty} \frac{1}{V_L} \frac{1}{e^{\beta(\varepsilon_s-\mu_L(\beta,\rho))} - 1} = 0\ ,$$

and $\lim_{L\to\infty} \rho_{(1,1,1)}(\beta,\rho) = \rho - \rho_m(\beta) > 0$, the *type* I vdBLP-GC.



2.11 For $\rho > \rho_m(\beta)$ there is a *coexistence* of the *saturated* **type III** vdBLP-GC, with the **constant** density $\rho_m(\beta) - \rho_c(\beta)$, and the standard BEC (the **type I** vdBLP-GC) in the the ground state with the density $\rho - \rho_m(\beta)$.


1.0
0.8
0.6
0.4
0.2
2. × 10⁻⁸
4. × 10⁻⁸
6. × 10⁻⁸
8. × 10⁻⁸
1. × 10⁻⁷

3. Exponential BEAM and CIGAR Traps

3.1 Remark 3.1 It is curious to note that neither **Casimir shaped boxes** $\Lambda = L^{\alpha_1} \times L^{\alpha_2} \times L^{\alpha_3}$, nor the **van den Berg boxes** $\Lambda = Le^{\alpha L} \times L \times L$, with **one-dimensional anisotropy** do not produce the *second* critical density $\rho_m(\beta) \neq \rho_c(\beta)$.

3.2 Remark 3.2 (BEAM) For beams with two critical densities we consider the Hamiltonian: $T_\Lambda^{(N=1)} = -\hbar^2\Delta/(2m) + m\omega_1^2 x_1^2/2$, with **harmonic trap** in direction x_1 and Dirichlet boundary conditions in directions x_2, x_3. Then the spectrum:

$$\left\{ \epsilon_s := \hbar\omega_1(s_1 + 1/2) + \frac{\hbar^2}{2m} \sum_{j=2}^{3} (\pi s_j / L_j)^2 \right\}_{s \in \mathbb{N}},$$

$s = (s_1, s_2, s_3) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}^2$, the ground-state energy: $\epsilon_{(0,1,1)}$.

3.3 For $\mu_L(\beta,\varrho) := \epsilon_{(0,1,1)} - \Delta_L(\beta,\varrho)$, the $\Delta_L(\beta,\varrho) \geq 0$ is a unique solution of the equation:

$$\varrho := \sum_{s=(s_1,1,1)} \omega_1 \frac{N_s(\beta,\mu)}{L_2 L_3} + \sum_{s\neq(s_1,1,1)} \omega_1 \frac{N_s(\beta,\mu)}{L_2 L_3},$$

$N_s(\beta,\mu) = (e^{\beta(\epsilon_s-\mu)} - 1)^{-1}$, $\omega_1 := \hbar/(mL_1^2)$ and $L_2 = L_3 = L$.

3.4 Similar to SLAB, for any $s_1 \geq 0$ and $\mu \leq 0$

$$\varrho(\beta,\mu) := \lim_{L_1,L\to\infty} \sum_{s\neq(s_1,1,1)} \omega_1 \frac{N_s(\beta,\mu)}{L_2 L_3} =$$

$$\frac{1}{(2\pi)^2}\int_0^\infty dp \int_{\mathbb{R}^2} \frac{d^2k}{e^{\beta(\hbar p + \hbar^2 k^2/2m - \mu)} - 1} \ .$$

The **first** critical density is *finite*: $\varrho_c(\beta) := \sup_{\mu\leq 0} \varrho(\beta,\mu) = \varrho(\beta,\mu=0) < \infty$.

3.5 For $\varrho > \varrho_c(\beta)$ the limit $L \to \infty$ of the first sum in **3.3**

$$\lim_{L_1, L \to \infty} \sum_{s=(s_1,1,1)} \omega_1 \frac{N_s(\beta, \mu_L)}{L_2 L_3} =$$

$$\lim_{L \to \infty} \frac{1}{L^2} \int_0^{\infty} \frac{dp}{e^{\beta(\hbar p + \Delta_L(\beta, \varrho))} - 1} =$$

$$\lim_{L \to \infty} \frac{1}{\hbar \beta L^2} \ln[\beta \Delta_L(\beta, \varrho)]^{-1} = \varrho - \varrho_c(\beta).$$

This gives the asymptotics : $\Delta_L(\beta, \varrho) = \beta^{-1} \, e^{-\hbar\beta(\varrho - \varrho_c(\beta)) L^2} + \ldots$.

3.6 Let $L_1 := Le^{\gamma L^2}$, $\gamma > 0$. Then, similar to SLAB, the representation of the limit in **3.5** by the integral is valid for $\hbar\beta(\varrho - \varrho_c(\beta)) \leq 2\gamma$ and we reach to necessity of the **second** critical density $\varrho_m(\beta) := \varrho_c(\beta) + 2\gamma/(\hbar\beta)$.

3.7 The rest of scenario is identical to the case of the SLAB.

3.8 Remark 3.3 (CIGAR) A "cigar"-type geometry is ensured by the anisotropic harmonic trap:

$$T_{\Lambda}^{(N=1)} = -\hbar^2\Delta/(2m) + \sum_{1\le j\le 3} m\omega_j^2 x_j^2/2 \ .$$

with $\omega_1 = \hbar/(mL_1^2),\ \omega_2 = \omega_3 = \hbar/(mL^2)$. Here $L_1, L_2 = L_3 = L$ are the *characteristic* sizes of the trap in three directions and the spectrum $\eta_s = \sum_{1\le j\le 3}\hbar\omega_j(s_j + 1/2)$.

3.9 For $\mu_L(\beta,n) := \eta_{(0,0,0)} - \Delta_L(\beta,n)$ and factor $\kappa > 0$:

$$\lim_{L_1,L\to\infty} \sum_{s=(s_1,0,0)} \kappa^3\omega_1\omega_2\omega_3 N_s(\beta,\mu_L) =$$

$$\lim_{L\to\infty} \frac{\kappa^3\hbar}{\beta(mL^2)^2} \ln[\beta\Delta_L(\beta,n)]^{-1} = n - n_c(\beta).$$

3.10 Again the **first** critical density $n_c(\beta) := n(\beta, \mu = 0)$ is finite:

$$n(\beta,\mu) := \lim_{L_1, L \to \infty} \sum_{s \neq (s_1, 0, 0)} \kappa^3 \omega_1 \omega_2 \omega_2 N_s(\beta,\mu) =$$

$$\int_{\mathbb{R}^3{}_+} \frac{\kappa^3 d\omega_1 d\omega_2 d\omega_3}{e^{\beta[(\omega_1+\omega_2+\omega_3)-\mu]} - 1} \ ,$$

and asymptotics:

$$\Delta_L(\beta, n > n_c(\beta)) = \beta^{-1} \ e^{-\beta(n-n_c(\beta))m^2 L^4/(\hbar\kappa^3)} + \dots \ .$$

3.11 If $L_1 := Le^{\widehat{\gamma} L^4}$, $\widehat{\gamma} > 0$, then the **second** critical density:

$$n_m(\beta) := n_c(\beta) + (\widehat{\gamma}\hbar\kappa^3)/(\beta m^2) \ .$$

is defined by the standard argument of the energy level spacing.

3.12 Bose-condensation (CIGAR) For $n_c(\beta) < n < n_m(\beta)$ we obtain the *type* III vdBLP-GC, when *none* of the single-particle states are *macroscopically* occupied:

$$n_s(\beta, \rho) := \lim_{L\to\infty} \frac{\kappa^3 \omega_1\omega_2\omega_3}{e^{\beta(\eta_s - \mu_L(\beta,n))} - 1} = 0 \ .$$

Although for $n_m(\beta) < n$ there is a coexistence of the *type* III vdBLP-GC, with the saturated constant density $n_m(\beta) - n_c(\beta)$, and the standard BEC (*type* I vdBLP-GC) in the ground-state:

$$n - n_m(\beta) = \lim_{L\to\infty} \frac{\kappa^3 \omega_1\omega_2\omega_3}{e^{\beta(\eta_{(0,0,0)} - \mu_L(\beta,n))} - 1} > 0 \ .$$

4. Temperature Dependence of the Bose-Condensate

4.1 The *first* critical temperatures: $T_c(\rho)$, $\widetilde{T}_c(\rho)$ or $\widehat{T}_c(\rho)$ are well-known. For a given density ρ they verify the identities:

$$\rho = \rho_c(\beta_c(\rho)) \ , \ \varrho = \varrho_c(\widetilde{\beta}_c(\varrho)) \ , \ n = n_c(\widehat{\beta}_c(n)) \ ,$$

respectively for slabs, squared beams or "cigars".

4.2 Since $\rho_c(\beta) =: T^{3/2}\, I_{sl}$, $\varrho_c(\beta) =: T^2\, I_{bl}$, $n_c(\beta) =: T^3\, I_{cg}$, the expressions for the **second** critical densities one gets relations between the *first* and the *second* critical temperatures:

$$\begin{aligned} T_m^{3/2}(\rho) + \tau^{1/2}\, T_m(\rho) &= T_c^{3/2}(\rho) \quad \text{(slab)} \ , \\ \widetilde{T}_m^2(\varrho) + \widetilde{\tau}\, \widetilde{T}_m(\varrho) &= \widetilde{T}_c^2(\varrho) \quad \text{(beam)} \ , \\ \widehat{T}_m^3(n) + \widehat{\tau}^2\, \widehat{T}_m(n) &= \widehat{T}_c^3(n) \quad \text{(cigar)} \ . \end{aligned}$$

$\tau = [\alpha m k_B/(\pi\hbar^2 I_{sl})]^2$, $\widetilde{\tau} = 2\gamma k_B/(\hbar I_{bl})$, $\widehat{\tau} = [(\widehat{\gamma}\hbar\kappa^3 k_B)/(m^2 I_{cg})]^{1/2}$ are "effective" temperatures related to the corresponding geometrical shapes.

4.3 Since the **total** condensate density is $\rho - \rho_c(\beta) := \rho_0(\beta) = \rho_{0c}(\beta) + \rho_{0m}(\beta)$, where $\rho_{0m}(\beta) := (\rho - \rho_m(\beta))\ \theta(\rho - \rho_m(\beta))$, the *second* critical temperature modifies the usual law for the condensate fractions temperature dependence.

4.4 For the *type* III vdBLP-GC, $\rho_{0c}(\beta)$, in the SLAB geometry:

$$\frac{\rho_{0c}(\beta)}{\rho} = \begin{cases} 1 - (T/T_c)^{3/2}\ , & T_m \le T \le T_c\ , \\ \sqrt{\tau}\ T/T_c^{3/2} & , \quad T \le T_m\ . \end{cases}$$

For the BEC (*type* I vdBLP-GC) in the ground state $\rho_{0m}(\beta)$:

$$\frac{\rho_{0m}(\beta)}{\rho} = \begin{cases} 0\ , & T_m \le T \le T_c, \\ 1 - (T/T_c)^{3/2}(1 + \sqrt{\tau/T}), & T \le T_m, \end{cases}$$

The **total** condensate density $\rho_0(\beta) := \rho_{0c}(\beta) + \rho_{0m}(\beta)$ results from *coexistence* of both of them: this gives the standard PBG expression $\rho_0(\beta)/\rho = 1 - (T/T_c)^{3/2}$.

4.5 For the "cigars" geometry the *type* III vdBLP-GC $r_{0c}(\beta)$:

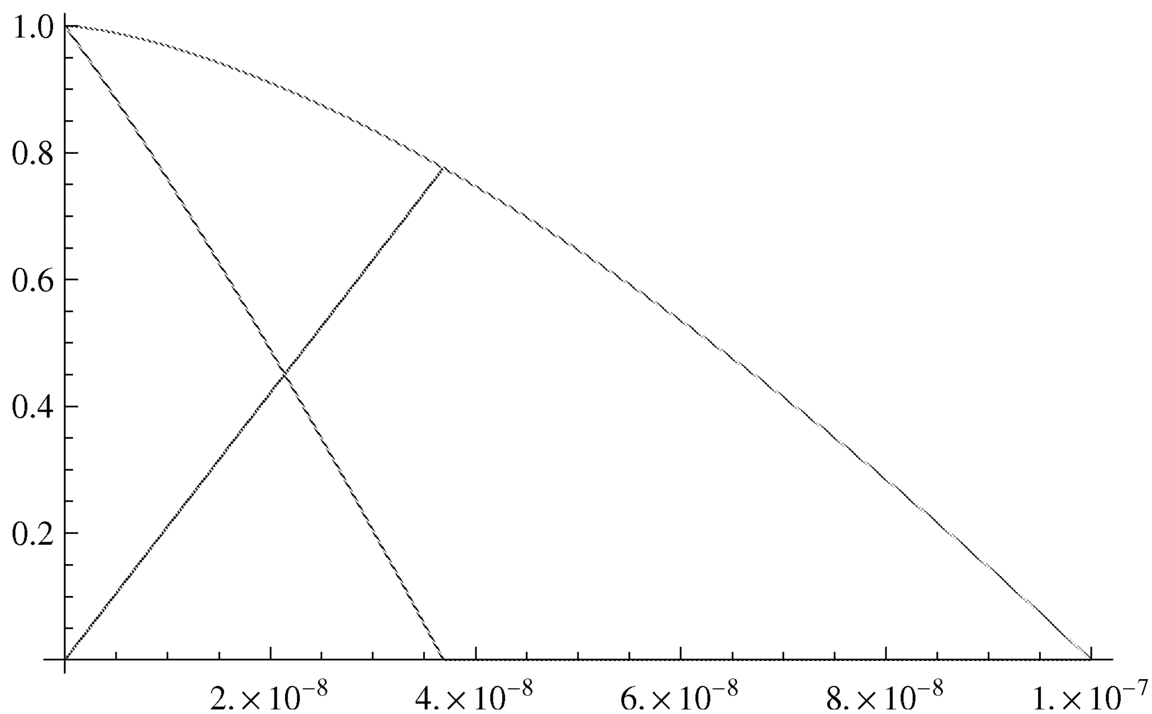
$$\frac{n_{0c}(\beta)}{n} = \begin{cases} 1 - (T/\widehat{T}_c)^3 \ , & \widehat{T}_m \leq T \leq \widehat{T}_c \ , \\ \widehat{\tau}^2 \ T/\widehat{T}_c^3 \ , & T \leq \widehat{T}_m \ . \end{cases}$$

The ground state conventional BEC is

$$\frac{n_{0m}(\beta)}{n} = \begin{cases} 0 \ , & \widehat{T}_m \leq T \leq \widehat{T}_c, \\ 1 - (T/\widehat{T}_c)^3(1 + \widehat{\tau}^2/T^2), & T \leq \widehat{T}_m, \end{cases}$$

and again for the two *coexisting* condensates one gets a standard expression:

$$n - n_c(\beta) := n_0(\beta) = n_{0c}(\beta) + n_{0m}(\beta) = (1 - (T/T_c)^{3/2})n \ .$$


1.0
0.8
0.6
0.4
0.2
2. × 10^−8
4. × 10^−8
6. × 10^−8
8. × 10^−8
1. × 10^−7

5. Anisotropy and Localisation

5.1 Global Scaled Particle Density :

$$\xi_L(u) := \sum_s \frac{|\phi_{s,\Lambda}(L_1u_1, L_2u_2, L_3u_3)|^2}{e^{\beta(\varepsilon_s-\mu)} - 1} \ ,$$

with the scaled distances $\{u_j = x_j/L_j \in [0,1]\}_{j=1,2,3}$.

5.2 For a given ρ in the slab geometry

$$\xi^{slab}_{\rho,L}(u) := \sum_s \frac{1}{e^{\beta(\varepsilon_s-\mu_L(\beta,\rho))} - 1} \prod_{j=1}^{d=3} \frac{2}{L_j}[\sin(\pi s_j u_j)]^2.$$

Since $2[\sin(\pi s_j u_j)]^2 = 1 - \cos\{(2\pi s_j/L_j)u_j L_j\}$ and $\lim_{L\to\infty} \mu_L(\beta, \rho < \rho_c(\beta)) < 0$, by the Riemann-Lebesgue lemma we obtain that $\lim_{L\to\infty} \xi^{slab}_{\rho,\Lambda}(u) = \rho$ for any $u \in (0,1)^3$.

5.3 If $\rho > \rho_c(\beta)$, then for any $u \in (0,1)^3$:

$$\lim_{L\to\infty} \sum_{s=(s_1,s_2,1)} \frac{1}{e^{\beta(\varepsilon_s-\mu_L(\beta,\rho))}-1} \prod_{j=1}^{d=3} \frac{2}{L_j}[\sin(\pi s_j u_j)]^2$$

$$= \lim_{L\to\infty} \frac{2[\sin(\pi u_3)]^2}{(2\pi)^2 L} \int_{\mathbb{R}^2} \frac{\prod_{j=1}^{2}(1-\cos(2k_j u_j L_j)d^2k}{e^{\beta(\hbar^2k^2/2m+\Delta_L(\beta,\rho))}-1}$$

$$= (\rho - \rho_c(\beta))\ 2[\sin(\pi u_3)]^2\ ,$$

$$\lim_{L\to\infty} \sum_{s\neq(s_1,s_2,1)} \frac{1}{e^{\beta(\varepsilon_s-\mu_L(\beta,\rho))}-1} \prod_{j=1}^{d=3} \frac{2}{L_j}[\sin(\pi s_j u_j)]^2$$

$$= \rho_c(\beta))$$

$$\Rightarrow \quad \xi_\rho^{slab}(u) = (\rho - \rho_c(\beta))\ 2[\sin(\pi u_3)]^2 + \rho_c(\beta)\ ,$$

which manifests a *space anisotropy* of the type III vdBLP-GC for $\rho_c(\beta) < \rho < \rho_m(\beta)$ in direction u_3.

5.4 For $\rho > \rho_m(\beta)$ one has to use representations and asymptotics from **2**. Then

$$\xi_\rho^{slab}(u) = (\rho - \rho_m(\beta)) \prod_{j=1}^{3} 2[\sin(\pi u_j)]^2 + (\rho_m(\beta) - \rho_c(\beta))\, 2[\sin(\pi u_3)]^2 + \rho_c(\beta) \ .$$

So, the anisotropy of the space particle distribution is still only in direction u_3 due to the type III vdBLP-GC ("quasi-condensate") $(\rho_m(\beta) - \rho_c(\beta))$. The input of the standard type I vdBLP-GC (one mode BEC) $(\rho - \rho_m(\beta))$ is **isotropic**.

6. Coherence Length and Anisotropy

6.1 ODLRO kernel:

$$K(x,y) := \lim_{L\to\infty} K_\Lambda(x,y) = \lim_{L\to\infty} \sum_s \frac{\overline{\phi}_{s,\Lambda}(x)\phi_{s,\Lambda}(y)}{e^{\beta(\varepsilon_s - \mu_L(\beta,\rho))} - 1} \ .$$

Let us center the box Λ at the origin of coordinates: $x_j = \tilde{x}_j + L_j/2$ and $y_j = \tilde{y}_j + L_j/2$. Then the **ODLRO** kernel gets the form:

$$K_\Lambda(\tilde{x},\tilde{y}) = \sum_{l=1}^{\infty} e^{l\beta\mu_L(\beta,\rho)} \ R_l^{(2)} \ R_l^{(1)} \ .$$

6.2 Here after the shift of coordinates and using additive form of the spectrum we put

$$R_l^{(2)}(\tilde{x}^{(2)}, \tilde{y}^{(2)}) = \sum_{s=(s_1,s_2)} e^{-l\beta\varepsilon_{s_1,s_2}} \, \overline{\phi}_{s_1,s_2,\Lambda}(\tilde{x}_1, \tilde{x}_2) \; \phi_{s_1,s_2,\Lambda}(\tilde{y}_1, \tilde{y}_2)$$

$$R_s^{(1)}(\tilde{x}_3, \tilde{y}_3) = \sum_{s=(s_3)} e^{-l\beta\varepsilon_{s_3}} \, \sqrt{\frac{2}{L_3}} \sin(\frac{\pi s_3}{L_3}(\tilde{x}_3 + \frac{L_3}{2})) \times \sqrt{\frac{2}{L_3}} \sin(\frac{\pi s_3}{L_3}(\tilde{y}_3 + \frac{L_3}{2})) \; .$$

6.3 By the Weyl theorem one gets for the first two directions:

$$\lim_{L\to\infty} R_l^{(2)}(\tilde{x}^{(2)}, \tilde{y}^{(2)}) = \frac{1}{l\lambda_\beta^2} \; e^{-\pi\|\tilde{x}^{(2)} - \tilde{y}^{(2)}\|^2/l\lambda_\beta^2} \; .$$

6.4 For exponentially anisotropic box and for $\rho_c(\beta) < \rho < \rho_m(\beta)$ we must split the sum over $s = (s_1, s_2, s_3)$ in **6.1** into two parts: sum over $s = (s_1, s_2, 1)$ and the rest. For the first sum by **6.3** we obtain:

$$\lim_{L\to\infty} \sum_{l=1}^{\infty} e^{l\beta\mu_L(\beta,\rho)} \sum_{s=(s_1,s_2,1)} e^{-l\beta\varepsilon_{s_1,s_2,1}} \times$$
$$\times \overline{\phi}_{s_1,s_2,1\Lambda}(\tilde{x})\ \phi_{s_1,s_2,1\Lambda}(\tilde{y}) =$$
$$\lim_{L\to\infty} \sum_{l=1}^{\infty} e^{-l\beta\Delta_L(\beta,\rho)} \frac{1}{l\lambda_\beta^2}\ e^{-\pi\|\tilde{x}^{(2)}-\tilde{y}^{(2)}\|^2/l\lambda_\beta^2} \times$$
$$\times \frac{2}{L} \sin(\frac{\pi}{L}(\tilde{x}_3 + \frac{L}{2}))\sin(\frac{\pi}{L}(\tilde{y}_3 + \frac{L}{2}))\ .$$

6.5 For the second part we apply the Weyl theorem for 3 component function:

$$\lim_{L\to\infty}\sum_{l=1}^{\infty} e^{l\beta\mu_L(\beta,\rho)} \sum_{s\neq(s_1,s_2,1)} e^{-l\beta\varepsilon_s}\times$$
$$\times\overline{\phi}_{s,\Lambda}(\tilde{x})\ \phi_{s,\Lambda}(\tilde{y}) = \sum_{l=1}^{\infty}\frac{1}{l\lambda_\beta^3}\ e^{-\pi\|\tilde{x}-\tilde{y}\|^2/l\lambda_\beta^2}\ .$$

6.6 Since $\Delta_L(\beta,\rho_c(\beta) < \rho < \rho_m(\beta)) \to 0, L \to \infty$, the change $l \to l\ \Delta_L(\beta,\rho)$ in **6.4** gives the integral Darboux-Riemann sum, where $\|\tilde{x}^{(2)} - \tilde{y}^{(2)}\|^2$ is scaled as $\|\tilde{x}^{(2)} - \tilde{y}^{(2)}\|^2\ \Delta_L(\beta,\rho)$.
6.7 Definition 6.1 The *coherence length* L_{ch} in direction perpendicular to x_3 is $L_{ch}(\beta,\rho)/L := \Delta_L^{-1/2}(\beta,\rho)$. A similar argument is valid for $\rho > \rho_m(\beta)$ with obvious modifications due to BEC for $s = (1,1,1)$ and adapted asymptotics for $\Delta_L(\beta,\rho)$.

6.7 To compare $L_{ch}(\beta,\rho)$ with the *scale* $L_{1,2} = Le^{\alpha L}$, we define the critical exponent $\gamma(T,\rho)$ such that

$$\lim_{L\to\infty} (L_{ch}(\beta,\rho)/L)(L_1/L)^{-\gamma(T,\rho)} = 1$$

Then

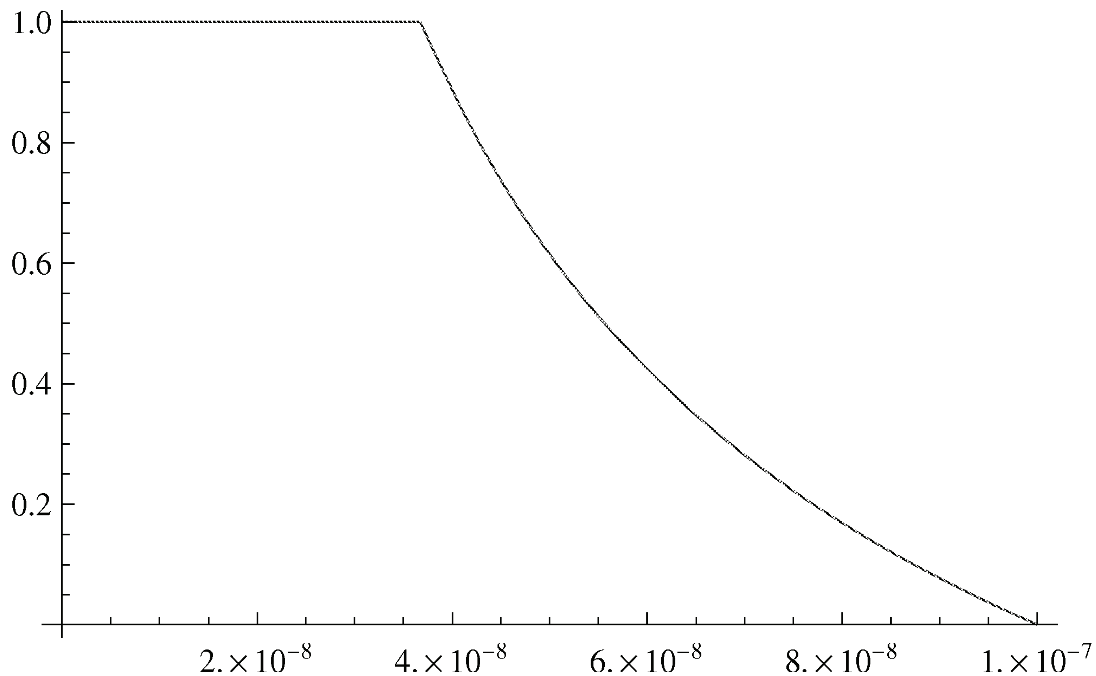
$$\begin{aligned}\gamma(T,\rho) &= \lambda_\beta^2\ (\rho - \rho_c(\beta))/2\alpha\ ,\ \ \rho_c(\beta) < \rho < \rho_m(\beta) \\ &= \lambda_\beta^2\ (\rho_m(\beta) - \rho_c(\beta))/2\alpha\ ,\ \ \rho_m(\beta) \le \rho\ .\end{aligned}$$

For a fixed density, taking into account temperature dependence of condensates we find the temperature dependence of the exponent $\gamma(T) := \gamma(T,\rho)$, see Fig:

$$\begin{aligned}\gamma(T) &= \sqrt{T/\tau}\ \{(T_c/T)^{3/2} - 1\}\ ,\ \ T_m < T < T_c\ , \\ &= 1,\ \ T \le T_m\ . \end{aligned} \tag{3}$$

6.8 Notice that in the both cases the ODLRO kernel is anisotropic due to the type III condensation in the states $s = (s_1, s_2, 1)$, whereas the other states give a symmetric part of correlations, which includes a constant term $\rho_c(\beta)$.

6.9 Numerically, for $L_1 = L_2 = 100\mu m$, $L_3 = 1\mu m$ and $T_m < T = 0.75 T_c$ the coherence length of the condensate is equal to $2.8\mu m \ll 100\mu m$. This decreasing of the *coherence length* for $T_c < T < T_m$ is experimentally observed (2003).


1.0
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0.2
$2.\times 10^{-8}$
$4.\times 10^{-8}$
$6.\times 10^{-8}$
$8.\times 10^{-8}$
$1.\times 10^{-7}$

Ch.3 BOSON RANDOM POINT PROCESSES and BEC

1. Random Point Processes (RPP)

• Let E be a locally compact metric space serving as the state-space of the *point* configurations $\xi \subset E$. By $\mathfrak{B}$ we denote the corresponding Borel σ-algebra on E and by $\mathfrak{B}_0 \subseteq \mathfrak{B}$ the *relatively* compact Borel sets in E. We denote by μ a *diffusive* (i.e. $\mu(x) = 0$ for any one-element subset $x \in E$) locally finite reference measure on $(E, \mathfrak{B})$. (The standard example is the Lebesgue measure $\mu(dx) = dx$ on $(E = \mathbb{R}^d, \mathfrak{B})$.)

We denote by Q_E the subspace of *locally-finite* point configurations $\{\xi \subset E\}$:

$$Q_E := \{\xi \subset E : \text{ card}(\xi \cap \Lambda) < \infty \text{ for all } \Lambda \in \mathfrak{B}_0\} \ .$$

• For any $\Lambda \in \mathfrak{B}_0$ one can define a subspace of the point configurations $Q_\Lambda := \{\xi \in Q_E : \xi \subset \Lambda\}$ and the mapping $\pi_\Lambda : \xi \mapsto \xi \cap \Lambda$ for the corresponding projection from Q_E onto Q_Λ. Then *counting* function: $N_\Lambda : \xi \mapsto \text{card}(\pi_\Lambda(\xi))$ is finite for any $\Lambda \in \mathfrak{B}_0$.

• Now one can introduce the notion of the *spatial* random point process (RPP) on $\mathbb{R}^d$ as locally finite discrete random sets $\xi \subset \mathbb{R}^d$, i.e. such that $N_\Lambda(\xi) < \infty$ for $\Lambda \in \mathfrak{B}_0$. Since below we use the Laplace transformation for characterisation of the RPPs, we need a more elaborated general setting.

• Let δ_x denote the atomic measure on $\mathfrak{B}$ supported at one-element subset $x \in E$. Then any configuration of points $\xi \in Q_E$ can be *identified* with the non-negative integer-valued Radon measure: $\lambda_\xi(\cdot) := \sum_{\{x \in \xi\}} \delta_x(\cdot)$ on the Borel σ-algebra $\mathfrak{B}$. Hence, $\lambda_\xi(D) = N_D(\xi)$ is the number of points that fall into the set $D \in \mathfrak{B}_0$ for the locally finite point configuration $\xi \in Q_E$.

• Recall that $C_0(E)^*$, which is dual to the space of continuous on E functions $C_0(E)$ vanishing at infinity and equipped with the uniform norm, is isometric (by the Riesz representation theorem) to the space $\mathcal{M}(E)$ of Radon measures on E. By this isometry the weak-$*$ topology on $C_0(E)^*$ yields the *vague* topology on $\mathcal{M}(E)$. Then identification of $\mathcal{M}(E)$ with the set of Radon measures λ_ξ induces on the point configuration space Q_E a topology, turning Q_E into a locally compact separable metric space with the corresponding Borel σ-algebra $\mathfrak{B}(Q_E)$. Note that if $\mathfrak{F}$ is the smallest σ-algebra on Q_E such that the mappings N_Λ are measurable for all $\Lambda \in \mathfrak{B}_0$, then $\mathfrak{F} = \mathfrak{B}(Q_E)$.

Definition. A random point process is a triplet $(Q_E, \mathfrak{B}(Q_E), \nu)$, where ν is a probability measure on $(Q_E, \mathfrak{B}(Q_E))$. Its marginal on Q_Λ is defined by the probability measure $\nu_\Lambda := \nu \circ \pi_\Lambda^{-1}$.

Note that the process defined above is *simple*, i.e. the random measure λ_ξ almost surely assigns measure ≤ 1 to *singletons*.

2. Correlation Functions and Laplace Transformation.

• For the marginal measure ν_Λ we consider the *Janossy* probability densities $\{j_{\Lambda,s}(x_1,\ldots,x_s)\}_{s\geq 0}$. Here $j_{\Lambda,s=0}(\emptyset) = \nu_\Lambda(\{\xi : N_\Lambda(\xi) = 0\})$ and for $s \geq 1$ it is a joint probability distribution that there are *exactly* s points in Λ, each located in the vicinity of the one of $x_1,\ldots,x_s$, and no points elsewhere. By construction the Janossy probability densities are *symmetric* and verify the *normalization* condition

$$\sum_{s=0}^{\infty} \frac{1}{s!} \int_{\Lambda^s} \mu(dx_1)\ldots\mu(dx_s)\; j_{\Lambda,s}(x_1,\ldots,x_s) = 1 \ ,$$

with a standard *convention* for $s = 0$.

• For any measurable function F on Q_Λ with components $\{F_s\}_{s\geq 0}$ one gets:

$$\int_{Q_\Lambda} \nu_\Lambda(d\xi) F(\xi) = \sum_{s=0}^{\infty} \frac{1}{s!} \int_{\Lambda^s} \mu(dx_1)\dots\mu(dx_s)$$
$$j_{\Lambda,s}(x_1,\dots,x_s)\; F_s(x_1,\dots,x_s)\ .$$

These joint probability distributions (*correlation functions*) serve for a very useful characterization of RPPs by the Laplace transformation.

• Let $f : E \to \mathbb{R}_+$, be non-negative continuous function with compact support. For each f one can define by

$$\langle f, \xi \rangle := \int_E \lambda_\xi(dx) f(x) = \sum_{x \in \xi} f(x) \ ,$$

the measurable function: $\xi \mapsto \langle f, \xi \rangle$ on Q_E. Then the Laplace transformation of the measure ν_Λ for a given f takes the form

$$\mathbb{E}_{\nu_\Lambda}(e^{-\langle f,\xi \rangle}) = \int_{Q_\Lambda} \nu_\Lambda(d\xi) \ e^{-\langle f,\xi \rangle} =$$

$$\sum_{s=0}^{\infty} \frac{1}{s!} \int_{\Lambda^s} \ \mu(dx_1) \ldots \mu(dx_s) \ j_{\Lambda,s}(x_1, \ldots, x_s) \ \prod_{j=1}^{s} e^{-f(x_j)} \ .$$

• The most fundamental example of RPP is the **Poisson point process** $\pi_z(d\xi)$ on $E = \mathbb{R}^d$ with Lebesgue measure $\mu(dx) = dx$ and *non-constant intensity* function $z(x) \geq 0$. For this RPF its marginal on Q_Λ is defined by the Janossy probability densities:

$$\{j_{\Lambda,s}(x_1, \ldots, x_s) = e^{-\int_\Lambda dx\, z(x)} \prod_{j=1}^{s} z(x_j)\}_{s \geq 0} \ .$$

Then one gets for any non-negative continuous function f with compact support the corresponding Laplace transformation (*generating* functional) expressed by the well-known formula:

$$\mathbb{E}_{\pi_z}(e^{-\langle f, \xi \rangle}) = \exp\left\{- \int_{\mathbb{R}^d} dx\, z(x)(1 - e^{-f(x)})\right\} \ ,$$

for extension to infinite configurations $Q_{\mathbb{R}^d}$.

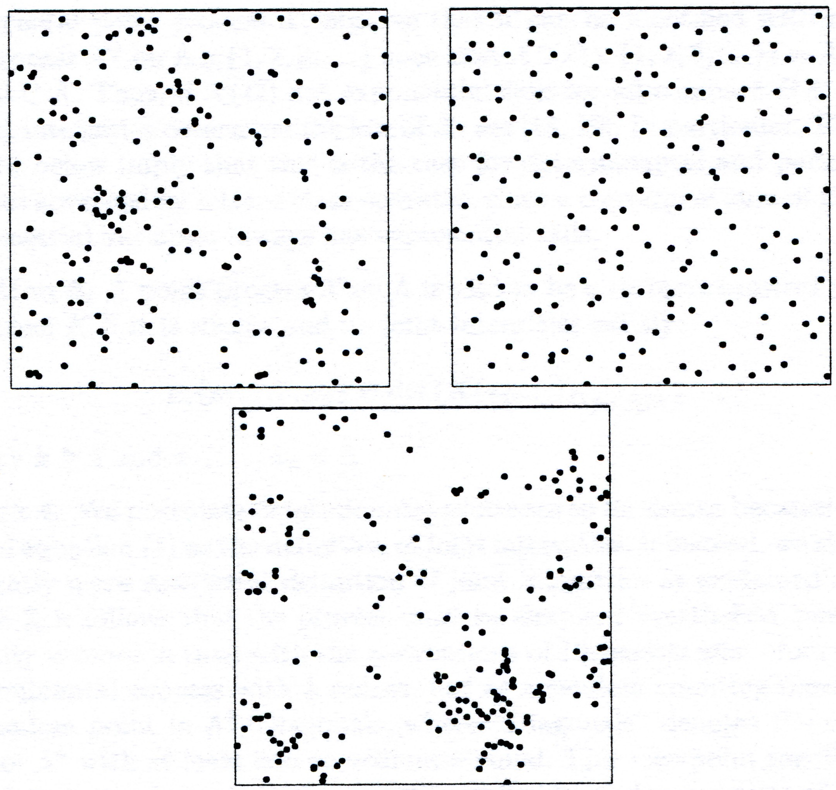


FIG 1. *Samples of translation invariant point processes in the plane: Poisson (left), determinantal (center) and permanental for* $K(z,w) = \frac{1}{\pi}e^{z\overline{w}-\frac{1}{2}(|z|^2+|w|^2)}$. *Determinantal processes exhibit repulsion, while permanental processes exhibit clumping.*

Example(continuation): The *Poisson* RPP π_λ with a *constant* intensity $\lambda \geq 0$

1. For any set $D \subset E$ of finite measure $\nu(D)$ one has:

$$\mathbb{P}\{N_D = n\} = \int_{Q(E)} \pi_\lambda(d\xi)\ \delta_{n,N_D(\xi)} = \frac{(\lambda\nu(D))^n}{n!}\ e^{-\lambda\nu(D)}\ .$$

2. For *mutually disjoint* subsets $\{D_n \subset \Lambda\}_{n\geq 1}$ the *Poisson* RPP π_λ has the properties:

$$\mathbb{P}\{\omega \in \Omega : \mu_\lambda^\omega(D) = n\} = \frac{(\lambda\nu(D))^n}{n!}\ e^{-\lambda\nu(D)}\ ,\ \ D \subset \Lambda\ ,$$

$$\mathbb{E}(\mu_\lambda^\omega(D_1)\ldots\mu_\lambda^\omega(D_k) = \lambda^k\nu(D_1)\ldots\nu(D_k) =$$

$$\mathbb{E}\mu_\lambda^\omega(D_1)\ldots\mathbb{E}\mu_\lambda^\omega(D_k).$$

• Recall that for any family of mutually *disjoint* subsets $\{D_n \subset \Lambda\}_{n\geq 1}$ the correlation functions (*joint intensities*) of the RPP μ^ω are defined by the densities $\{\rho_n : \Lambda^n \mapsto \mathbb{R}^1_+\}_{n\geq 1}$ with respect to the measure ν:

$$\mathbb{E}(\mu^\omega(D_1)\dots\mu^\omega(D_n)) = \int_{D_1\times\ldots\times D_n} \nu(dx_1)\dots\nu(dx_n)\ \rho_n(x_1,\dots,x_n)$$

• Let $K(x,y)$ be a kernel of non-negative self-adjoint *locally* Tr-class operator on $L^2(\Lambda)$.

Definition: A RPP is called determinantal/permanental with the kernel K, if it is simple and its correlation functions:

$$\rho_n(x_1,\dots,x_n) = \mathbf{det}\|K(x_i,x_j)\|_{1\leq i,j\leq n}$$
$$\rho_n(x_1,\dots,x_n) = \mathbf{per}\|K(x_i,x_j)\|_{1\leq i,j\leq n}\ .$$

For any $n\geq 1$ and $x_1,\dots,x_n\in\Lambda$. $\det_\alpha A := \sum_{\sigma\in\mathfrak{S}_n} \alpha^{n-c(\sigma)}\prod_{1\leq i\leq n} a_{i\sigma(i)}$
$\alpha = \pm 1 \Leftrightarrow$ per/ det and $c(\sigma)$ is the number of *cycles* in σ.

2. Fermion/Boson Random Point Processes

2.1 Quantum (Statistical) Mechanics: Fermions

• Let $\mathfrak{H}_L := L^2(\Lambda_L), \Lambda_L = [-L/2, L/2]^d$ and $\Delta_{L,p}$ be Laplacian with *periodic* boundary conditions on $\partial\Lambda_L$, i.e.

$$\text{spec}(-\Delta_{L,p}) = \{\varepsilon(k) = (2\pi/L)^2\|k\|^2 : k \in \mathbb{Z}^d\}.$$

The Gibbs semigroup kernel has the form:

$$(G_{\beta,L})(x,y) := (e^{\beta\Delta_L})(x,y) = \sum_{k\in\mathbb{Z}^d} e^{-\beta\varepsilon(k)}\phi_{k,L}(x)\overline{\phi_{k,L}(y)} =$$

$$\sum_{k\in\mathbb{Z}^d} (G_\beta)(x, y + kL),$$

where the "heat" semigroup kernel:

$$(G_\beta)(x,y) := \lim_{L\to\infty} (G_{\beta,L})(x,y) = (4\pi\beta)^{-d/2}\exp(-\|x-y\|^2/4\beta).$$

- **Remark:** Any n-particle free-fermion wave function is the *Slater* determinant:

$$\Psi_{k_1,\ldots,k_n}(x_1,\ldots,x_n) = \frac{1}{\sqrt{n!}} \det \|\phi_{k_i,L}(x_j)\|_{1\le i,j\le n}$$

- The corresponding n-point free-fermion joint *probability distribution* density: $p_{n,L}(x_1,\ldots,x_n) := |\Psi_{k_1,\ldots,k_n}(x_1,\ldots,x_n)|^2$, or

$$p_{n,L}(x_1,\ldots,x_n) = \frac{1}{n!} \det \|\phi_{k_i,L}(x_j)\|_{1\le i,j\le n} \overline{\det \|\phi_{k_i,L}(x_j)\|_{1\le i,j\le n}}$$

- Since $\det A \det B = \det A\, B$ one gets:

$$p_{n,L}(x_1,\ldots,x_n) = \frac{1}{n!} \det \|K_{n,L}(x_i,x_j)\|_{1\le i,j\le n}\,,$$

where $K_{n,L}(x,y) = \sum_{1\le i\le n} \phi_{k_i,L}(x)\overline{\phi_{k_i,L}(y)}$ is the kernel of orthogonal projection on the Env$\{\phi_{k_1,L},\ldots,\phi_{k_n,L}\}$.

- Since the k-point *marginal* correlation functions are

$$p^{(k)}_{n,L}(x_1,\dots,x_n) := \frac{n!}{(n-k)!}\int p_{n,L}(x_1,\dots,x_n)dx_{k+1},\dots,dx_n$$
$$= \det \|K_{n,L}(x_i,x_j)\|_{1\le i,j\le k} \ ,$$

the determinantal RPP $\mu^{\omega,F}_{n,L}$ generated by the joint probability distribution density $p_{n,L}$ is correctly defined for n free fermions in the cube Λ_L.

- **Canonical Ensemble:** Probability density distribution of n free-fermion positions in the cube Λ_L:

$$p_{n,L}(x_1,\dots,x_n;\beta) := Z^{-1}_{\Lambda,F}(\beta,n) \times$$
$$\times \sum_{(k_1,\dots,k_n)\in(\mathbb{Z}^n)} \overline{\Psi_{k_1,\dots,k_n}(x_1,\dots,x_n)} \left(\bigotimes^n G_{\beta,L}\Psi_{k_1,\dots,k_n}\right)(x_1,\dots,x_n).$$

• **Proposition:** Let $(x_1, \ldots, x_n) \mapsto \xi := \sum_{1\le j\le n} \delta_{x_j} \in Q(\Lambda_L)$. Then $p_{n,L}(x_1, \ldots, x_n; \beta)$ induces a determinantal RPP $\mu^{\omega,F}_{\beta,n,L}$ with matrix $K_{\beta,n,L}(x_i, x_j) = (G_{\beta,L})(x_i, x_j)$, i.e. a probability measure $d\mu^F_{\beta,n,L}(\xi)$ on the configuration space $Q(\Lambda_L)$.

• **Laplace Transformation:** Let $\langle \xi, f\rangle := \sum_{1\le j\le n} f(x_j)$, where *non-negative* $f \in C_0(\Lambda_L)$. Then for $\widetilde{G}_{\beta,L} := \sqrt{G_{\beta,L}}\; e^{-f} \sqrt{G_{\beta,L}}$:

$$\mathbb{E}_{\beta,n,L}(e^{-\langle \xi,f\rangle}) := \int_{Q(\Lambda_L)} d\mu^F_{\beta,n,L}(\xi)\; e^{-\langle \xi,f\rangle} =$$

$$\int_{\Lambda_L^n} dx_1 \ldots dx_n\; p_{n,L}(x_1, \ldots, x_n; \beta) \exp\{-\sum_{1\le j\le n} f(x_j)\} =$$

$$\int_{\Lambda_L^n} dx_1 \ldots dx_n \det \|(\widetilde{G}_{\beta,L})(x_i, x_j)\| / \int_{\Lambda_L^n} dx_1 \ldots dx_n \det \|(G_{\beta,L})(x_i, x_j)\|.$$

- **Example:**(recall) For the Poisson RPP one obtains:

$$\int_{Q(\Lambda_L)} d\mu_\lambda(\xi)\ e^{-\langle \xi, f\rangle} = \exp[-\int_{\Lambda_L} dx\ \lambda(1 - e^{-f(x)})]\ .$$

• **Thermodynamic Limit:**[Shirai-Takahashi ('03)] For $n/L^d \to \rho$ a weak limit of the RPP: $w - \lim_{L\to\infty} \mu^F_{\beta,n,L} = \mu^F_{\beta,\rho}$, exists and

$$\int_{Q(\mathbb{R}^d)} d\mu^F_{\beta,\rho}(\xi)\, e^{-\langle \xi, f\rangle} = \mathrm{Det}[I - \sqrt{1-e^{-f}} z_* G_\beta (I + z_* G_\beta)^{-1} \sqrt{1-e^{-f}}]$$

$$\rho = \int_{\mathbb{R}^d} \frac{d^d q}{(2\pi)^d} \frac{z_* e^{-\beta\|q\|^2}}{1 + z_* e^{-\beta\|q\|^2}} = (z_* G_\beta (I + z_* G_\beta)^{-1})(x,x).$$

• For a Tr-class integral operator J on $L^2(\Lambda,\lambda)$, the **Fredholm determinant/permanent** (Vere-Jones' formula ('88)):

$$\mathrm{Det}([I - \alpha J]^{-1/\alpha}) = \sum_{s=0}^{\infty} \int_{\Lambda^s} \lambda^{\otimes s}(dx_1 \ldots dx_n) \mathrm{det}_\alpha \| J(x_i, x_j) \|_{1 \le i,j \le n},$$

where $\mathrm{det}_{\alpha=\pm 1} = \mathrm{per}/\,\mathrm{det}$.

2.2 Quantum (Statistical) Mechanics: Bosons

• (Grand -) Canonical Ensemble:

Probability density distribution of n free-boson positions in the cube Λ_L:

$$p_{n,L}(x_1,\ldots,x_n;\beta) := Z^{-1}_{\Lambda,B}(\beta,n)\times$$

$$\times \sum_{(k_1,\ldots,k_n)\in(\mathbb{Z}^n)} \overline{\Psi_{k_1,\ldots,k_n}(x_1,\ldots,x_n)} \left(\bigotimes^n G_{\beta,L}\Psi_{k_1,\ldots,k_n}\right)(x_1,\ldots,x_n),$$

$$\Psi_{k_1,\ldots,k_n}(x_1,\ldots,x_n) = \frac{1}{\sqrt{n!\prod_l n(k_l)!}}\ \mathrm{per}\|\phi_{k_i,L}(x_j)\|_{1\le i,j\le n}$$

• The boson RPP $d\mu^{B}_{\beta,n,L}(\xi)$ on the configuration space $Q(\Lambda_L)$ is implied by $p_{n,L}$. In the (*grand* -) canonical thermodynamic limit for particle densities $\rho < \rho_c(\beta)$ (or solutions $z_*(\beta, \rho) < 1$), where:

$$\rho = \int_{\mathbb{R}^d} \frac{d^d q}{(2\pi)^d} \frac{z_* e^{-\beta\|q\|^2}}{1 - z_* e^{-\beta\|q\|^2}} = (z_* G_\beta (I - z_* G_\beta)^{-1})(x, x) < \rho_c(\beta)$$

one obtains [Tamura-Ito, ('06)]:

$$\int_{Q(\mathbb{R}^d)} d\mu^B_{\beta,\rho}(\xi)\, e^{-\langle \xi, f\rangle} = \mathrm{Det}[I + \sqrt{1-e^{-f}}\, z_* G_\beta (I - z_* G_\beta)^{-1} \sqrt{1-e^{-f}}]^{-1}$$

• **Proposition:**[Tamura-Ito, ('07)] For $\rho > \rho_c(\beta)$ one obtains $z_* = 1$ and

$$\int_{Q(\mathbb{R}^d)} d\mu^B_{\beta,\rho}(\xi)\, e^{-\langle \xi, f\rangle} = \frac{\exp[-(\rho-\rho_c(\beta))(\sqrt{1-e^{-f}}, [I+K_f]^{-1}\sqrt{1-e^{-f}})]}{\mathrm{Det}[I+K_f]},$$

where $K_f := \sqrt{1-e^{-f}}\, G_\beta (I - G_\beta)^{-1} \sqrt{1-e^{-f}}$ is from the Tr-class.

• The free boson RPP ($\rho > \rho_c(\beta)$) = a convolution of a boson RPP (at $z_* = 1$) and a boson processes (*numerator*) proportional to the condensate density: $\rho - \rho_c(\beta)$.

• **2.3 Grand-Canonical (β, μ) Free Bose-Gas without QM:**
(a) *Independent* random variables $k \mapsto N_k \in \mathbb{N} \cup \{0\}$, $k \in \Lambda^*{}_L$, in the probability space $\Omega := \times_{k\in\Lambda^*{}_L} \Omega_k$.
(b) For *bosons* the one-mode random *occupation* numbers are: $N_k \geq 0$ (for *fermions*: $N_k = 0, 1$).
(c) *Probabilities* (**N.B.** for *bosons*: $\mu < 0$, since $\varepsilon_k = \|k\|^2 \geq 0$) :

$$\mathrm{Pr}_{\beta,\mu}(N_k) := e^{-\beta(\varepsilon_k-\mu)N_k}/\Xi_k(\beta,\mu)\ ,\ k \in \Lambda^*{}_L.$$

(d) *Expectations* for $k \in \Lambda^*{}_L$, here $z_* := e^{-\beta\,\mu}$:

$$\mathbb{E}_{\beta,\mu}(N_k) = \frac{1}{e^{\beta(\varepsilon_k-\mu)} - 1}.$$

(e) Expectation value of the *total* density of bosons in $\mathbb{R}^d$:

$$\lim_{L\to\infty} \rho_{\Lambda_L}(\beta,\mu) := \lim_{L\to\infty} \frac{1}{|\Lambda_L|} \sum_{k\in\Lambda^*} \mathbb{E}_{\beta,\mu}(N_k) = \int_0^\infty \frac{d\tilde{N}_d(E)}{e^{\beta(E-\mu)} - 1}\ .$$

3. Bosons in a Weak Harmonic Trap [Tamura-Z.(2009)]

3.1 Weak Harmonic Trap

- One-particle Hamiltonian of harmonic oscillator:

$$h_\kappa = \frac{1}{2}\sum_{j=1}^{d}\left(-\frac{\partial^2}{\partial x_j^2} + \frac{x_j^2}{\kappa^2} - \frac{1}{\kappa}\right),$$

self-adjoint operators in the Hilbert space $\mathfrak{H} := L^2(\mathbb{R}^d)$, with

$$\mathrm{Spec}(h_\kappa) = \{\,\epsilon_\kappa(s) := |s|_1/\kappa \,|\, s = (s_1, \cdots, s_d) \in \mathbb{N}^d\,\}$$

$|s|_1 := \sum_{j=1}^{d} s_j$.

- In this setup the "thermodynamic limit" is an "opening" of the trap $\kappa \to \infty$: the Weak Harmonic Trap (WHT) limit.

- Perfect gas expectation value of *total* number of particles:

$$N_\kappa(\beta,\mu) = \frac{1}{\beta}\frac{\partial \ln \Xi_{0,\kappa}(\beta,\mu)}{\partial\mu} = \sum_{s\in\mathbb{N}^d} \frac{1}{e^{\beta(\epsilon_\kappa(s)-\mu)} - 1} \,. \qquad (4)$$

- Since $N_\kappa(\beta,\mu)$ diverges for $\kappa\to\infty$ as κ^d , the scaled density:

$$\rho_\kappa(\beta,\mu) := \frac{1}{\kappa^d}\sum_{s\in\mathbb{N}^d} \frac{1}{e^{\beta(\epsilon_\kappa(s)-\mu)} - 1} \,,$$

$$\rho(\beta,\mu) = \lim_{\kappa\to\infty}\rho_\kappa(\beta,\mu) = \int_{[0,\infty)^d} \frac{dp}{e^{\beta(|p|_1-\mu)} - 1} = \sum_{s=1}^{\infty} \frac{e^{\beta\mu s}}{(\beta s)^d} \,.$$

- **Integrated Density of States ($\mathcal{N}(E)$) and critical density:**

$$\mathcal{N}_{d,\kappa}(E) = \frac{1}{\kappa^d} \sum_{s \in \mathbb{N}^d} \theta(E - |s|_1/\kappa).$$

Then we obtain in the $\kappa \to \infty$ limit

$$d\mathcal{N}_d(E) = \frac{E^{d-1}}{\Gamma(d)} dE \neq \frac{E^{(d-2)/2}}{(2\pi)^{d/2}\Gamma(d/2)} dE = d\tilde{\mathcal{N}}_d(E)$$

$$\rho_c(\beta) := \zeta(d)/\beta^d \neq \zeta(d/2)/(2\pi\beta)^{d/2} =: \tilde{\rho}_c(\beta)$$

3.2 Mean-Field Interaction and Main Results

• The mean-field interacting bosons trapped in the harmonic potential is defined by its *grand-canonical partition function*

$$\Xi_{\lambda,\kappa}(\beta,\mu) := \sum_{n=0}^{\infty} e^{\beta(\mu n - \lambda n^2/2\kappa^d)} \mathrm{Tr}_{\mathfrak{H}^n_{symm}}[\otimes^n G_\kappa(\beta)] \ ,$$

$G_\kappa(\beta) = e^{-\beta h_\kappa}$ is Gibbs semigroup for oscillator process, $\beta > 0$, $\lambda > 0$ and arbitrary $\mu \in \mathbb{R}^1$.

• **Theorem:**(*Normal phase*) Let $\mu < \mu_{\lambda,c}(\beta)(:= \lambda\rho_c(\beta))$. Then the boson RPP $\nu_{\kappa,\beta,\mu}$ converges weakly in the WHT limit $\kappa \to \infty$ to the RPP ν_{β,r_*} with the Laplace transformation:

$$\mathbb{E}_{\beta,r_*}\left[e^{-\langle f,\xi\rangle}\right] = \mathrm{Det}\left[1 + \sqrt{1-e^{-f}}\; r_* G_\beta (1 - r_* G_\beta)^{-1} \sqrt{1-e^{-f}}\;\right]^{-1},$$

$r_* = r_*(\beta,\mu,\lambda) \in (0,1)$ is a unique solution of the equation :

$$\beta\mu = \ln r + \lambda\beta \int_0^\infty \frac{d\mathcal{N}_d(E)}{r^{-1}e^{\beta E} - 1}\,, \quad r := e^{\beta(\mu-\lambda\rho)} < 1\ .$$

• **Theorem:**(*Condensed phase*) For $\mu > \mu_{\lambda,c}(\beta)(:= \lambda\rho_c(\beta))$ the Laplace transformation of the boson RPP measure has the following limit:

$$\lim_{\kappa\to\infty} \frac{1}{\kappa^{d/2}} \ln \mathbb{E}_{\beta,\mu}\left[e^{-\langle f,\xi\rangle}\right] =$$
$$-\frac{\mu-\mu_{\lambda,c}(\beta)}{\pi^{d/2}\lambda}\left(\sqrt{1-e^{-f}},(1+K_f)^{-1}\sqrt{1-e^{-f}}\right),$$

where

$$K_f := \left(G_\beta^{1/2}(1-G_\beta)^{-1/2}\sqrt{1-e^{-f}}\right)^* \left(G_\beta^{1/2}(1-G_\beta)^{-1/2}\sqrt{1-e^{-f}}\right)$$

is a positive trace-class operator on $\mathfrak{H} = L^2(\mathbb{R}^d)$ for $d > 2$.

• **Remark:** (*Condensed phase*) Similar to the homogeneous free Bose-gas the resulting RPP is a *convolution* of **two** BoseRPP [Tamura-Z.(2009)].

3.3 Local Particle Density: $f \in C_0(\mathbb{R}^d)$ and $f \geq 0$

- **Corollary:**(*Normal phase*) For $\mu < \mu_{\lambda,c}(\beta)$

$$\mathbb{E}_{\beta,r_*}\Big[\langle f, \xi\rangle\Big] = \mathrm{Tr}[f\ r_* G(\beta)(1 - r_* G(\beta))^{-1}] = \rho_{r_*} \int_{\mathbb{R}^d} dx\ f(x)\ ,$$

where the local density ρ_{r_*} in the neighbourhoods of the bottom of the WHT potential is given by

$$\rho_{r_*} = r_* G(\beta)(1 - r_* G(\beta))^{-1}(x,x) = \sum_{n=1}^{\infty} r_*^n/(2\pi\beta n)^{d/2}.$$

- **Corollary:**(*Condensed phase*) For $\mu > \mu_{\lambda,c}(\beta)$,

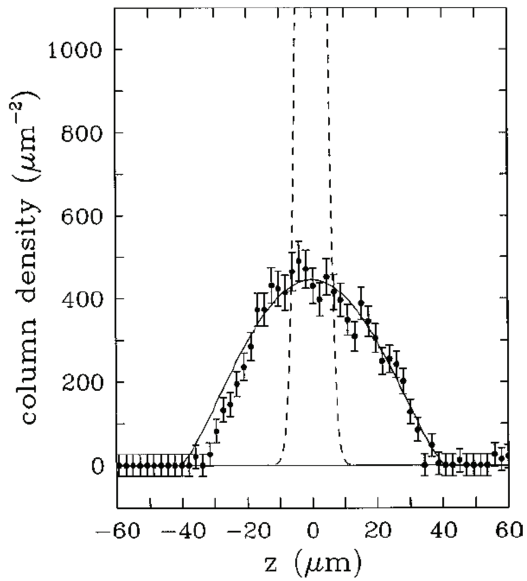
$$\liminf_{\kappa\to\infty} \frac{\mathbb{E}_{\kappa,\beta,\mu,\lambda}\Big[\langle f, \xi\rangle\Big]}{\kappa^{d/2}} \geqslant \frac{\mu - \mu_{\lambda,c}(\beta)}{\pi^{d/2}\lambda} \int_{\mathbb{R}^d} dx\ f(x)\ .$$

Density Profile

✓ Strong effect of interaction (repulsive interactions expand the condensed cloud typically of a factor between 2 and 10)

$$n_{\mathrm{TF}}(\mathbf{r}) \simeq \frac{m}{4\pi\hbar^2 a}\left[\mu - \frac{1}{2} m\omega_i^2 r_i^2\right]$$

➡ bimodal distribution (two-component, condensed and normal, cloud)



[F. Dalfovo *et al*. *Rev. Mod. Phys*. **71**, 463 (1999)]

3.4 Global Particle Density:

• The results of Theorem and Corollary in the non-condensed regime has the following interpretation: in the WHT limit the position distribution of the MF interacting bosons in neighbourhoods of the **origin** of coordinates (i.e. the bottom of the WHT potential) is close to that of a free Perfect BG corresponding to the *unconventional* parameter r^* (instead of conventional z^*). The information about the particle position distribution in domains distant from the bottom of the WHT is missing in the limit ν_{β,r_*} since f has a finite support.

• In order to take this "tail"-particles into account we have to use for our model the standard definition of the grand-canonical global number of particles :

$$\rho^{(tot)}_{\kappa,\lambda}(\beta,\mu) := \frac{1}{\kappa^d\,\beta}\frac{\partial \ln \Xi_{\kappa}(\beta,\mu)}{\partial \mu}$$

$$= \frac{1}{\kappa^d\,\Xi_{\kappa,\lambda}(\beta,\mu)}\sum_{n=0}^{\infty} n\, e^{\beta(\mu n-\lambda n^2/2\kappa^d)}\ \mathrm{Tr}_{\mathfrak{H}^n_{symm}}[\otimes^n G_{\kappa}(\beta)]\ .$$

• Since κ^d is interpreted as the effective volume of the model, the function $\rho_{\kappa,\lambda}^{(tot)}(\beta,\mu)$ represents an effective total space-averaged density of the non-homogeneous boson gas.

• **Theorem:**(*global density* = *experiment*) In the WHT limit

$$\rho_{\lambda}^{(tot)}(\beta,\mu) = \lim_{\kappa\to\infty} \rho_{\kappa,\lambda}^{(tot)}(\beta,\mu) = \lim_{\kappa\to\infty} \kappa^{-d}\mathrm{Tr}[r_* G_\kappa (1 - r_* G_\kappa)^{-1}]$$

exists and satisfies:

(i) $\mu \leq \mu_{\lambda,c}(\beta)$:

$$\rho_{\lambda}^{(tot)}(\beta,\mu) = \int_0^\infty \frac{d\mathcal{N}_d(E)}{r_*^{-1} e^{\beta E} - 1} \quad \text{and} \quad \beta\mu = \log r_* + \lambda\beta\rho_{\lambda}^{(tot)}(\beta,\mu)\ ;$$

(ii) $\mu > \mu_{\lambda,c}(\beta)$: $(\rho_c^{(tot)}(\beta) := \lim_{\mu\to\mu_c(\beta)} \rho_{\lambda}^{(tot)}(\beta,\mu) = \zeta(d)/\beta^d)$

$$\rho_{\lambda}^{(tot)}(\beta,\mu) = \mu/\lambda = (\mu - \mu_{\lambda,c}(\beta))/\lambda + \rho_c^{(tot)}(\beta)\ .$$

4 Conclusion: Bosons in a Weak Harmonic Trap

• Different behaviour of the space distributions of bosons described in the Theorems has the following explanation:
In the *normal case* the bosons are distributed almost uniformly in the region of radius κ according to the shape of the oscillator process kernel .

• On the other hand, in the *condensed phase case* the condensed part of particles $\kappa^d(\rho_\lambda^{(tot)}(\beta,\mu) - \rho_{\lambda,c}^{(tot)}(\beta)) = \kappa^d(\mu - \mu_{\lambda,c}(\beta))/\lambda$ is localized in the region of radius $O(\kappa^{1/2})$ according to profile of the square of the ground state wave function

$$\Omega_\kappa(x) = \frac{1}{(\pi\kappa)^{d/4}} e^{-\|x\|^2/2\kappa} \equiv \phi_{s=0,\,\kappa}(x).$$

Whereas the particles outside of the condensate essentially spread out over the region of radius κ.

5 Large Deviation Principle for non-interacting BRPP

5.1 Non-Interacting BRPP with BEC

• **Proposition:**[Tamura-Ito, ('07)] For continuous $f \geq 0$ with compact $supp$ we define two BRPP by generating functionals:

$$\int_{Q(\mathbb{R}^d)} d\mu^{(det)}_{K,z}(\xi) e^{-\langle f,\xi\rangle} = \det[1+K_f(z)]^{-1} \ , \ \ z = e^{\beta\mu} \leq 1 \, , \qquad (1)$$

$$\int_{Q(\mathbb{R}^d)} d\mu_{K,\varrho}(\xi) e^{-\langle f,\xi\rangle} = \exp\{-\varrho(\sqrt{1-e^{-f}}, \frac{1}{1+K_f(1)}\sqrt{1-e^{-f}})\}, (2)$$

where $K_f(z) := \sqrt{1-e^{-f}} z G_\beta (1+zG_\beta)^{-1} \sqrt{1-e^{-f}}$ and $G_\beta) := e^{\beta\Delta}$. Then the BRPP for the *ideal* gas is $\mu^B_{K,\rho\leq\rho_c} = \mu^{(det)}_{K,z\leq 1}$, but in the regime of BEC $(\rho > \rho_c)$ it is the *convolution*:

$$\mu^B_{K,\rho>\rho_c} := \mu^{(det)}_{K,z=1} * \mu_{K,\varrho=\rho-\rho_c} = (\text{non} - \text{Condensate}) * (\text{Condensate})$$

- **Theorem (LLN)**[Tamura-Z. ('09)] For continuous function $f \geq 0$ with compact $supp$, the limit

$$\lim_{\kappa \to \infty} \frac{1}{\kappa^d} \langle f(\cdot/\kappa), \xi \rangle = \rho \int_{\mathbb{R}^d} dx\, f(x) \ ,$$

holds in $L^2(Q(\mathbb{R}^d), \mu^B_{K,\rho})$.

- **Theorem (CLT)**[Tamura-Z. ('09)] Let $\rho > \rho_c$. Then for $\kappa \to \infty$ the family of random variables

$$X_\kappa := \frac{\langle f(\cdot/\kappa), \xi \rangle - \rho \kappa^d \int_{\mathbb{R}^d} f(x)\, dx}{\sqrt{2(\rho - \rho_c)}\ \|(-\beta \Delta)^{-1/2} f\|_{HS}\ \kappa^{(d+2)/2}}$$

converges in distribution to the standard Gaussian random variable:

$$\lim_{\kappa \to \infty} \int_{Q(\mathbb{R}^d)} d\mu^B_{K,\rho>\rho_c}(\xi) e^{itX_\kappa} = e^{-t^2/2}$$

6.2 Large Deviation Principle in the BEC regime

• **Theorem (LDP)**[Tamura-Z. ('09)] For $\rho > \rho_c$ there exists a convex rate function $I(s) := \sup_{s\in\mathbb{R}}\Big(st - P(t)\Big)$, such that:

$$\limsup_{\kappa\to\infty} \frac{1}{\kappa^{d-2}} \log \mu^B_{K,\rho}\Big(\frac{1}{\kappa^d}\Big\langle f\Big(\cdot/\kappa\Big), \xi\Big\rangle \in F\Big) \leqslant -\inf_{s\in F} I(s)\ ,\ \text{for closed } F \subset \mathbb{R}\ ,$$

and

$$\liminf_{\kappa\to\infty} \frac{1}{\kappa^{d-2}} \log \mu^B_{K,\rho}\Big(\frac{1}{\kappa^d}\Big\langle f\Big(\cdot/\kappa\Big)\ ,\ \xi\Big\rangle \in G\Big) \geqslant -\inf_{s\in G} I(s)\ ,\ \text{for open } G \subset \mathbb{R}\ .$$

$$P(t) = \lim_{\kappa\to\infty} \frac{1}{\kappa^{d-2}} \log \int_{Q(\mathbb{R}^d)} d\mu^B_{K,\rho}(\xi) e^{\frac{t}{\kappa^2}\langle f(\,\cdot/\kappa),\xi\rangle} = P^{(det)}_{K,z=1}(t) + P_{K,\rho-\rho_c}(t)$$

$$\begin{cases} t\rho_c \int_{\mathbb{R}^d} f(x)\,dx + (\rho-\rho_c)t^2(f,(-\beta\Delta - tf)^{-1}f) & t < \|\sqrt{f}(-\beta\Delta)^{-1}\sqrt{f}\|^{-1} \\ +\infty & t \geq \|\sqrt{f}(-\beta\Delta)^{-1}\sqrt{f}\|^{-1} \end{cases}$$

6.3 BEC versus the normal phase

• Let $D_\kappa := \langle f(\cdot/\kappa), \xi\rangle/\kappa^d$ be the a *random empirical density* of particles localized in the region of length scale κ.

For the BEC case $\rho > \rho_c$:

(i) The random variable D_κ converges for $\kappa \to \infty$ to its expectation value $m := \rho \int_{\mathbb{R}^d} f(x)\, dx$ in mean.

(ii) The law of the random variable $\kappa^{(d-2)/2}(D_\kappa - m)$ converges to normal distribution as $\kappa \to \infty$.

(iii) The law of the random variable D_κ manifests a large deviation property with parameter κ^{d-2}.

• **For the normal phase** $\rho \le \rho_c$:

(i) also holds;

(ii) holds but for $\kappa^{d/2}(D_\kappa - m)$, instead of $\kappa^{(d-2)/2}(D_\kappa - m)$;

(iii) holds with the order κ^d, instead of κ^{d-2}.

END

THANK YOU FOR YOUR ATTENTION !

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