Assignment 1

Problem 1. Show that the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$$

is not absolutely convergent. Show by example that, indeed, by permuting the terms of this series one can obtain series with different limits.

Problem 2. Give an ϵ-δ proof for

$$\sum_{n=0}^{\infty}x^n = \frac{1}{1-x} \quad \text{for all } x \in (-1,1)$$

Problem 3. Suppose you know that $\mathbb{R}$ with its usual metric is a complete metric space.

a) Prove that $\mathbb{R}^n$ is complete with the usual Euclidean metric.

b) Is $\mathbb{R}^n$ also complete with the following metrics, derived from the sum and maximum norm respectively:

$$\begin{aligned} d_{\text{sum}}(x,y) &= |x_1-y_1|+|x_2-y_2|+\cdots|x_n-y_n| \\ d_{\max}(x,y) &= \max\{|x_1-y_1|,|x_2-y_2|,\ldots|x_n-y_n|\} \end{aligned}$$

Why (not) ?

Problem 4. Let (x_n) and (y_n) be two bounded sequences in $\mathbb{R}$.

a) Prove that

$$\limsup_n (x_n+y_n) \le \limsup_n x_n + \limsup_n y_n$$

b) Show, by giving an example, that the inequality in a) can be strict.

Problem 5. Let $a_{n,m} \in [-1, 1]$ for all $n, m = 1, 2, \ldots$, be such that the following limits exist:

$$\begin{aligned} \lim_{n\to\infty} a_{n,m} &= x_m \quad \text{for all } m, \quad \lim_{m\to\infty} a_{n,m} = y_n \quad \text{for all } n \\ \lim_{m\to\infty} x_m &= x, \quad \lim_{n\to\infty} y_n = y \end{aligned}$$

a) Does one necessarily have $x = y$? Give a proof or a counterexample.

b) What about the same question with all lim's replaced by sup 's?

Problem 6. For $r > 0, x_0 \in \mathbb{R}^n$, Show that the closed ball defined by $\overline{B}_r(x_0) = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$ is indeed, as the notation suggests, the closure of the open ball $B_r(x_0) = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$

Problem 7. Prove that, in any metric space, the intersection of two open sets is open.