Assignment 2

(due date: Monday, Oct 13)

Problem 1. Consider the function $f : (0, +\infty) \to \mathbb{R}$ defined by $f(x) = \sin(1/x)$.

a) Is f bounded? (prove or disprove)

b) Is f continuous on $(0, +\infty)$? (prove or disprove)

c) Can f be extended to $[0, +\infty)$ by defining its value in 0 in an appropriate way, such that f is continuous on $[0, +\infty)$? Why (not)?

d) Answer the same questions a-c) for the function g defined by $g(x) = xf(x)$.

Problem 2. A function $f : [a, b] \to \mathbb{R}$ is called *convex* on $[a, b]$ if for all $x, y \in [a, b]$, and $t \in [0, 1]$ one has

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

Show that if f is convex on $[a, b]$, then f is upper semi-continuous (but not necessarily continuous) on $[a, b]$.

Problem 3. Let (X, d) be a metric space, and let (f_n) be a (not necessarily convergent) sequence of continuous real-valued functions on a common domain $D \subset (X, d)$. Define a function f on D by

$$f(x) = \inf_n f_n(x)$$

for all $x \in D$.

a) Prove that f is upper semi-continuous.

b) Show by example that f is not necessarily continuous.

Problem 4. Let (X, d) be a metric space and $f : X \to \mathbb{R}$ a continuous function. Show that for any open $G \subset \mathbb{R}$, $f^{-1}(G) \subset X$ is open. (For any $A \subset \mathbb{R}$, $f^{-1}(A)$ is by definition the set $\{x \in X \mid f(x) \in A\}$.)