Assignment 4

(due date: Monday, Oct 27)

The problems in this assigment belong together. The solution of any one of them involves information from of one or more of the previous ones.

Problem 1. Let V be a real linear space, and $(\cdot,\cdot): V \times V \to \mathbb{R}$ a non-negative bilinear form on V, i.e., for all $x, y, z \in V$ and $a, b, \in \mathbb{R}$ one has the relations:
i) $(ax + by, z) = a(x, z) + b(y, z)$ (linear)
ii) $(x, y) = (y, x)$ (symmetric)
iii) $(x, x) \geq 0$ (non-negative)
Prove the *Cauchy-Schwartz* inequality:

$$(x, y)^2 \leq (x, x)(y, y) \quad \text{for all } x, y \in V$$

[Hint: consider $(ax + by, ax + by)$ as a function of a and b.]

Problem 2. Let $f \in C([a, b])$. Prove that

$$\left| \int_a^b f(x)\, dx \right| \leq |b - a|^{1/2} \left(\int_a^b f(x)^2\, dx \right)^{1/2}$$

[Hint: apply the Cauchy-Schwartz inequality to the appropriate bilinear form on $C([a, b])$.]

Problem 3. Consider a subset A of $C([a, b])$ defined as follows:

$$A = \{f \in C([a, b]) \mid f' \in C([a, b]), f(a) = f(b) = 0, \text{ and } \textstyle\int_a^b f'(x)^2\, dx \leq M\}$$

where M is an arbitrary real constant. Prove that A is precompact in $(C([a, b]), \|\cdot\|_{\sup})$.

Problem 4. Consider functions $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} a_n \sin(n\pi x) \qquad (*)$$

for all $x \in [0, 1]$, where the a_n are real numbers satisfying $\sum_{n=0}^{\infty} |a_n| < +\infty$.

a) Prove that $f \in C([0,1])$.

b) Prove that the set of all f of the form (*) that satisfy $\|f\|_{\text{sup}}$, is not precompact in $(C([0,1]), \|\cdot\|_{\text{sup}})$.

c) Define a set $A \subset C([0,1])$ as follows:

$$A = \{f \in C([0,1]) \mid f \text{ is of the form } (*) \text{ and } \textstyle\sum_{n=0}^{\infty} n^2 a_n^2 \leq 1\}$$

Prove that A is precompact in $(C([0,1]), \|\cdot\|_{\text{sup}})$.