Assignment 7

(due date: Monday, Nov 24)

Problem 1. Consider the space of functions

$$V = \{f : [0, 2\pi] \to \mathbb{C} \mid f \text{ is continuous}\}$$

as a linear space over $\mathbb{C}$ with the usual operations of pointwise addition of functions and multiplication by a scalar. Prove that the set $A \subset V$ defined by

$$A = \{f \in V, f(x) = e^{inx} \mid n = 0, 1, 2, \ldots\}$$

is linearly independent.

Problem 2. Let V be a linear space over $\mathbb{R}$. A subset $C \subset V$ is called *convex* if for all $x, y \in C$ and $t \in \mathbb{R}, 0 \leq t \leq 1$,

$$tx + (1 - t)y \in C$$

a) Prove that if C_1 and C_2 are two convex subsets of V, then so is $C_1 \cap C_2$.
b) Show that $C_1 \cup C_2$ need not be convex even if C_1 and C_2 are.
c) For two subsets $A, B \subset V$, the set $A + B$ is defined by

$$A + B = \{x + y \mid x \in A, y \in B\}$$

Prove that for any two convex sets C_1 and C_2, the set $C_1 + C_2$ is also convex.

Problem 3. Prove that a norm on a linear space is a convex function.

Problem 4. Let X be the space of all sequences $x = (x_1, x_2, x_3, \ldots)$ of complex numbers and define

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \min\{1, |x_n - y_n|\}$$

a) Prove that d defines a metric on X.

b) Prove that (X, d) is a complete metric space.

c) Is X a Banach space?