Assignment 8

(due date: Wednesday, December 3)

Problem 1. Let X and Y be normed linear spaces and (T_n) a sequence of elements in $\mathcal{B}(X,Y)$, converging uniformly to $T \in \mathcal{B}(X,Y)$. Prove that $\lim_{n\to\infty} T_n x = Tx$ for all $x \in X$, i.e., prove that (T_n) converges to T in the strong operator topology.

Problem 2. Consider $T \in \mathcal{B}(X)$. Prove that $\lambda\mathbb{1} - T$ is invertible for all $\lambda \in \mathbb{C}$, such that $|\lambda| > \|T\|$, and that the inverse is a bounded linear map. (Hint: use the Neumann series to define $(\lambda\mathbb{1} - T)^{-1}$.)

Problem 3. Consider the Banach space l_2 defined by

$$l_2 = \{z = (z_n)_{n=1}^\infty \mid z_n \in \mathbb{C}, \sum_{n=1}^\infty |z_n|^2 < +\infty\}$$

and

$$\|z\| = \sqrt{\sum_{n=1}^\infty |z_n|^2}$$

For $k = 1, 2, \ldots$, define $(e_{k,n})_{n=1}^\infty := e_k \in l_2$ by

$$e_{k,n} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{else} \end{cases}$$

and define $P_k : l_2 \to l_2$, $P_k z = z_k e_k$, for all $z \in l_2$.

a) Prove that $P_k \in \mathcal{B}(l_2)$, for all $k \geq 1$, and that $P_k P_l = \delta_{k,l} P_k$ ($\delta_{k,l}$ is the Kronecker delta : $\delta_{k,l} = 0$, if $k \neq l$, and $\delta_{k,k} = 1$, for all k, l.)

b) Define $S_k \in \mathcal{B}(l_2)$ by

$$S_k = \sum_{l=1}^k P_l$$

Prove that $S_k^2 = S_k$, and $\|S_k\| = 1$.

c) Prove that $S_k \to \mathbb{1}$ in the strong operator topology.

d) Does S_k converge to $\mathbb{1}$ uniformly? Why (not) ?