9 Examples of Hilbert spaces in applications

9.1 Quantum Mechanics

In the 20's Quantum Mechanics developed at a feverish pace, and people did hardly have time to pause and think about the mathematical structures they were using and/or developing. Only later was it realized (by John von Neumann) that Hilbert space is the natural setting for Quantum Mechanics. But the harm had been done: generations of physicists have been educated in the "new mechanics" without any knowledge of the proper language, which added even more mystery to the already quite mysterious theory.

One of the simplest mechanical systems where quantum mechanics is needed for an accurate description is a particle, confined to move on a straigth line and bouncing back and forth between two parallel walls (the so-called particle in a box). Quantum effects are important when its kinetic energy is not much larger than $m\hbar^2/L^2$, where m is the mass of the particle, $\hbar$ is Planck's constant, and L is the distance between the walls.

In Quantum Mechanics the state of the particle at each instant in time t is described by an element $\psi(t,\cdot) \in L^2([0,L])$, i.e., a vector in the Hilbert spaces of square-integrable complex-valued funtions on the interval $[0,L]$. This is in constrast to classical mechanics à la Newton, where the state is described by just two numbers: the position of the particle ($0 \leq x \leq L$) and its velocity ($v \in \mathbb{R}$).

The dynamics of the system is determined by a PDE, called the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t), \quad x \in [0,L], t \in \mathbb{R} \tag{9.1}$$

with the boundary conditions $\psi(0,t) = \psi(L,t) = 0$, for all $t \in \mathbb{R}$.

We will discuss this equation and its solutions lateron, making use of Hilbert space notions.

9.2 Natural L^2-spaces

L^2-spaces often arise naturally in many context because

$$\int |f(x)|^2\, dx \tag{9.2}$$

is a fundamental quantity of the model that one wants to be finite. E.g.,

- $f(x)$ is the velocity of a fluid at the point with coordinates x. Then, for an incompressible fluid, $\int_V |f(x)|^2\, dx$ is the total kinetic energy of the fluid, which, for any finite volume V, should be finite.

- $f(x)$ is a random variable with vanishing mean. The space of all such random variables with finite variance is an L^2-space.

- The electromagnetic field in space-time is described by two vector fields $E(x,t)$ and $B(x,t)$. $|E(x,t)|^2 + |B(x,t)|^2$ (in suitable units) is the energy density of the electromagnetic field, and hence, again, an integral of the form (9.2) is a fundamental quantity in the theory: the total energy.

9.3 Sobolev spaces

Sometimes, the relevant quantity is not just the L^2 norm but rather a combination of the L^2-norm and the L^2-norm of some derivatives. On $C^n([a,b]$, the space of functions with n continuous derivatives, one defines the *Sobolev* norms by

$$\|f\|_{n,2} = \sum_{k=0}^{n} \int_a^b |f^{(k)}(x)|^2\, dx \tag{9.3}$$

where $f^{(k))}$ denotes the k-th derivative of f. The Hilbert space obtained by completion of $C^n([a,b]$ with the norm $\|\cdot\|_{n,2}$, is denoted by $H^n([a,b])$. These Hilbert spaces are called *Sobolev spaces*. They are widely used in the analysis of partial differential operators needed for PDE.

10 Special o.n. bases, Fourier series

What makes Hilbert space so powerful in applications is the possibility to express a problem in a suitable o.n. basis, e.g., one of the bases of orthogonal polynomials introduced introduced in Section 8.3. But orthogonal polynomials are by no means the only interesting o.n. bases. The most widely used is probably the Fourier basis, consisting of trigonometric functions, which is the subject of this chapter. Here, we will introduce this basis, which is the mathematical formulation underlying Fourier series, and investigate its general properties. Later, in Capter **??**, we will explore a variety of applications of Fourier series. In Chapter **??** we will give a brief introduction to yet another class of important special o.n. bases, the so-called *wavelets*.

10.1 Fourier series

Our main objective is to prove that $\{e_n \mid n \in \mathbb{Z}\}$ with

$$e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx} \tag{10.1}$$

is a o.n. basis of $L^2([0, 2\pi])$. Of course one can then construct an o.n. basis for $L^2([a, b])$, for any $a < b \in \mathbb{R}$, by simple rescaling.

In concreto, this then means that for any $f \in L^2([0, 2\pi])$, we have

$$f(x) = \sum_{n\in\mathbb{Z}} a_n \frac{1}{\sqrt{2\pi}} e^{inx} = \frac{a_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} (a_n \frac{a_0}{\sqrt{2\pi}} e^{inx} + a_{-n} \frac{a_0}{\sqrt{2\pi}} e^{-inx})$$

where the equality means convergence of the partial sums to f in the L^2-norm.

Our proof consists of the following steps:

1. $\{e_n \mid n \in \mathbb{Z}\}$ is an orthonormal set:

$$\begin{aligned} \int_0^{2\pi} \overline{\frac{1}{\sqrt{2\pi}} e^{imx}} \frac{1}{\sqrt{2\pi}} e^{inx} \, dx &= \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)x} \, dx \\ &= \begin{cases} 1 & \text{if } n = m \\ \frac{e^{i(n-m)2\pi} - e^{i(n-m)0}}{i(n-m)} = 0 & \text{if } m \neq n \end{cases} \end{aligned}$$

2. Let P denote the set of all finite linear combinations of functions e_n. These are the *trigonometric polynomials*. According to Theorem 8.5 one of the equivalenty conditions for completeness is that P is dense in $L^2([0, 2\pi])$, i.e., $\overline{P} = [\{e_n\}] = L^2([0, 2\pi])$. By, construction the continuous functions $C([0, 2\pi])$ are dense in $L^2([0, 2\pi])$. Therefore, it is sufficient to show that P is dense in the continous functions with the L^2-norm.

3. We will in fact prove a stronger result, namely that P is dense in the *periodic* continuous functions (i.e, continuous f such that $f(0) = f(2\pi)$, or, equivalently, the continuous functions defined on the unit circle, which we denote by $C(T)$), with the *uniform* norm, which is a useful fact in its own right.

Exercise 10.1 *a) Prove that P is dense in the periodic continuous functions on $[0, 2\pi]$ with the uniform norm, implies P is dense in the space of all continous functions on $[0, 2\pi]$ with the L^2-norm.*
b) Is P dense in the space of all continous functions $[0, 2\pi]$ with the uniform norm?

Theorem 10.1 *The trigonometric polynomials are dense in $(C(T), \|\cdot\|_{\sup})$.*

Proof: We parametrize T by the interval $[0, 2\pi]$ with 0 and 2π identified. So, $C(T)$ is then the space of continuous function $[0, 2\pi] \to \mathbb{C}$, that satisfy $f(0) = f(2\pi)$.

For this proof we introduce the notion of *approximate identity.* We will define a sequence of operators $I_n : C(T) \to P$ such that

$$\lim_{n\to\infty} \|I_n f - f\|_{\sup} = 0, \qquad \text{for all } f \tag{10.2}$$

It is clear that such a sequence of operators approximating the identity operator in the uniform norm on the continuous functions. Obviously, if we can find such a sequence of operators the theorem follows.

We claim that the following I_n satisfy (10.2).

$$I_n f(x) = \int_0^{2\pi} \varphi_n(x-y) f(y)\, dy$$

where

$$\varphi_n(x) = c_n(1 + \cos x)^n$$

where the c_n are positive constants chosen such that

$$\int_0^{2\pi} \varphi_n(x)\, dx = 1, \qquad \text{for all } n \tag{10.3}$$

Note that $\varphi_n \in P$ for all n:

$$\varphi_n(x) = \sum_{k=-n}^{n} a_n e^{inx}$$

First of all, the I_n are well-defined maps $C(T) \to P$. Indeed, for any $f \in C(T)$ we have

$$I_n f(x) \;=\; \int_0^{2\pi} \sum_{k=-n}^{n} a_k e^{ik(x-y)} f(y)\, dy$$

$$= \sum_{k=-n}^{n} a_k e^{ikx} \int_0^{2\pi} e^{-iky} f(y)\, dy$$
$$= \sum_{k=-n}^{n} b_k e^{ikx}$$

where
$$b_k = a_k \int_0^{2\pi} e^{-iky} f(y)\, dy$$

Next, we have to check the uniform convergence (10.2). This is similar to the proof of Weierstrass approximation theorem. We use that any continous f on a compact (here T), is bounded and uniformly continuous. We also use (10.3), and note that
$$\int_0^{2\pi} \varphi_n(x-y) f(y)\, dy = \int_0^{2\pi} \varphi_n(y) f(x-y)\, dy$$
Pick $\epsilon > 0$ such that $|x-y| < \epsilon \Rightarrow |f(x) - f(y)| \le \delta$. Then

$$\begin{aligned}
|I_n f(x) - f(x)| &= \left| \int_0^{2\pi} \varphi_n(y)(f(x-y) - f(x))\, dy \right| \\
&= \left| \int_{|y|\le\epsilon} \varphi_n(y)(f(x-y) - f(x))\, dy \right| + \left| \int_{|y|>\epsilon} \varphi_n(y)(f(x-y) - f(x))\, dy \right| \\
&\le \delta \int_{|y|\le\epsilon} \varphi_n(y) + 2\|f\|_{\sup} \int_{|y|>\epsilon} \varphi_n(y)\, dy
\end{aligned}$$

By taking the sup over x, and by using
$$\lim_n \int_{|y|>\epsilon} \varphi_n(y)\, dy = 0, \qquad \text{for all } \epsilon > 0$$
we obtain (10.2). ∎

We have proved that for $f \in L^2([0, 2\pi])$
$$\lim_{N\to\infty} \|\frac{1}{2\pi} \sum_{n=-N}^{N} \left[\int_0^{2\pi} e^{-iny} f(y)\, dy \right] e^{inx} - f(x)\|_{L^2} = 0 \qquad (10.4)$$

Theorem 7.6 implies that (10.4) is the best possible approximation with a trigonometric polynomial, in the sense of L^2. It is interesting to ask for other kinds of convergence of the partial sums of the series to the function. E.g., it can be proved that they also converge pointwise a.e. to $f(x)$. The proof of this is beyond the scope of this course.

One can also show that for *smooth* f the convergence of the partial sums is *uniform* (see Section 10.2 for a more detailed result. Another interesting question is how the partial sums behave in the neighborhood of a point of discontinuity of f. See H. Dym, and H.P. McKean, *Fourier Series and Integrals*, Section 1.6 for a discussion of this question (the Gibbs Phenomenon).

It is sometimes useful to modify the o.n. basis $\{e_n(x)\}$ a little bit. It is clear that the functions

$$\begin{aligned} e_0(x) &= \frac{1}{\sqrt{2\pi}} \\ \frac{1}{\sqrt{2}}(e_n + e_{-n}) &= \frac{1}{\sqrt{\pi}}\cos(nx), \quad n \geq 1 \\ \frac{1}{i\sqrt{2}}(e_n - e_{-n}) &= \frac{1}{\sqrt{\pi}}\sin(nx), \quad n \geq 1 \end{aligned}$$

together, also form an o.n. basis. This is useful, e.g., because, in this basis, a real-valued function will have real Fourier coeffients. Another useful property of the o.n. basis with cos and sin functions is the following. Consider all functions to be defined on $[-\pi, \pi]$. A function f is called *even* if $f(-x) = f(x)$, for all x, and f is called *odd* if $f(-x) = -f(x)$, for all x. It is then easy to verify that even functions f have an expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx)$$

while for odd f

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx) \quad .$$

The Fourier coefficients of f are often denoted by $\hat{f}(n)$.

Exercise 10.2 *Prove that the sets* $\{e_n \mid n \geq 1\}$ *defined by*

$$e_n(x) = \sqrt{\frac{2}{\pi}}\sin nx, \quad 0 \leq x \leq \pi$$

and $\{f_n \mid n \geq 0\}$ *defined by*

$$f_0(x) = \sqrt{\frac{1}{\pi}}, \quad f_n(x) = \sqrt{\frac{2}{\pi}}\cos nx, \quad 0 \leq x \leq \pi, n \geq 1,$$

are two o.n. bases of $L^2([0, \pi])$.

10.2 Fourier series of $f \in C^p(T)$

Our aim is to show that if f has p continuous derivatives, with $p = 1, 2, 3, \ldots$, then the partial sums S_N of its Fourier series converge uniformly to f, and to give an estimate of the speed of convergence. We will use the following observation. If $f \in C^1(T)$, we can relate the Fourier coefficients of f' to the ones of f by integration by parts:

$$\begin{aligned} \widehat{f'}(n) &:= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} f'(x)\, dx \\ &= \frac{1}{\sqrt{2\pi}}[f(2\pi) - f(0)] - \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} (-in) e^{-inx} f(x)\, dx \\ &=: in\hat{f}(n) \end{aligned} \tag{10.5}$$

By induction it follows from from (10.5) that if $f \in C^p(T)$ then $\widehat{f^{(p)}}(n) = (in)^p \hat{f}(n)$.

Note that for relation (10.5) to hold it is, in fact, sufficient that $f' \in L^2(T)$. We will return to this later when we discuss weak derivatives and distributions.

Theorem 10.2 *If $f \in C^p(T)$ the partial sums*

$$S_N(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$$

satisfy

$$\|S_N - f\|_{\sup} \leq \frac{\|f^{(p)}\|_{L^2}}{p\sqrt{2\pi} N^{p-1/2}}$$

In particular, the partial sums converge uniformly to f.

Proof: Note that $f \in C(T)$ and $S_N \in C(T)$ for all N. Due to the completeness of $C(T)$ for the supremum norm it is, therefore, sufficient to prove that for all $M \geq N$

$$\|S_N - S_M\|_{\sup} \leq \frac{\|f^{(p)}\|_{L^2}}{p N^{p-1/2}}$$

For this we use (10.5) and the inequalities of Schwarz and Bessel:

$$\begin{aligned} \|S_N - S_M\|_{\sup} &\leq \frac{1}{\sqrt{2\pi}} \sum_{N<|n|\leq M} |\hat{f}(n)| \\ &= \frac{1}{\sqrt{2\pi}} \sum_{N<|n|\leq M} |\widehat{f^{(p)}}(n)| \frac{1}{|n|^p} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\sqrt{2\pi}}\left[\sum_{N<|n|\leq M}|\widehat{f^{(p)}}(n)|^2\right]^{1/2}\left[\sum_{N<|n|\leq M}\frac{1}{|n|^{2p}}\right]^{1/2} \\
&\leq \frac{1}{\sqrt{2\pi}}\|f^{(p)}\|_{L^2}\times\left[2\int_N^\infty \frac{1}{r^{2p}}\,dr\right]^{1/2} \\
&\leq \frac{\|f^{(p)}\|_{L^2}}{p\sqrt{2\pi}N^{p-1/2}}
\end{aligned}
$$

■

10.3 A Sobolev Embedding Theorem

As an application of the foregoing we discuss the simplest example of a Sobolev Embedding Theorem. Recall the we defined the Sobolev space $H^1(T)$ as the Hilbert space obtained by completion of $C^1(T)$ for the norm derived from the inner product

$$\langle f, g\rangle_{H^1} = \langle f, g\rangle_{L^2} + \langle f', g'\rangle_{L^2}$$

Using the the orthonormality of the Fourier basis and (10.5), we can rewrite this as

$$\langle f, g\rangle_{H^1} = 2\pi\sum_{n=-\infty}^{+\infty}\overline{\hat{f}(n)}\hat{g}(n) + \overline{\widehat{f'}(n)}\widehat{g'}(n)$$

In other words, $H^1(T)$ is also the completion of the trigonometric polynomials for the H^1 norm, and, therefore

$$H^1(T) = \{\sum_{n=-\infty}^{+\infty} a_n e^{inx} \mid \sum_{n=-\infty}^{+\infty} n^2|a_n|^2 < \infty\}$$

Theorem 10.3 (A Sobolev Embedding Theorem) *$H^p(T)\subset C^{p-1}(T)$.*

Proof: In the same way as in the proof of Theorem 10.2, one shows that the condition

$$\sum_{n=-\infty}^{+\infty} n^{2p}|a_n|^2 < \infty$$

implies that the partial sums for $f^{(p-1)}$ converge uniformily. And, hence, the limit is continous. ■

10.4 Multiple Fourier series

Let $n \geq 1$, and denote by T^n the $n-$dimensional torus, i.e., the Cartesian product of n circles. The notion of Fouries series is straighforwardly extended to functions of n variables $f : T^n \to \mathbb{C}$, by considering the o.n. basis of $L^2(T^n)$ consisting of the functions

$$e_{\mathbf{n}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} e^{i\mathbf{n}\cdot\mathbf{x}}, \mathbf{n} \in \mathbb{Z}^n$$

Where $\mathbf{x} = (x_1, \ldots, x_n) \in T^n$ and $\mathbf{n} = (n_1, \ldots, n_n) \in \mathbb{Z}^n$, and $\mathbf{n} \cdot \mathbf{x} = n_1 x_1 + \cdots + n_n x_n$.

Exercise 10.3 *Consider $f : T^3 \to \mathbb{C}$ defined by*

$$f(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} a_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{x}}$$

Where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{n} = (n_1, n_2, n_3)$, and $\mathbf{n} \cdot \mathbf{x} = n_1 x_1 + n_2 x_2 + n_3 x_3$. Prove that if, for some $\alpha > 3$,

$$\sum_{\mathbf{n}\in\mathbb{Z}^3} \|n\|^{\alpha} |a_{\mathbf{n}}|^2 < \infty$$

f is continuous.