11 Applications of Fourier series to PDE

The applications of Fourier series that we will cover this week are all elementary, but they demonstrate that Fourier series are a very useful tool in the study of a wide variety of concrete problems in engineering and science. There will be some loose ends here and there because we do not discuss conditions for existence and uniqueness of solutions of PDE problems.

11.1 The heat/diffusion equation

This is the original application for which Jean-Baptiste Fourier developed the series expansion named after him, but suggested earlier by Daniel Bernoulli (J.-B. Fourier, *Théorie analytique de la chaleur*, 1822).

11.1.1 Heat conduction in an insulated ring

Consider a thin ring made of a heat conducting material and modelled by the unit circle T. The temperature at time t and position x along the ring is given by $u(x,t)$, which satisfies the *heat equation* given by

$$u_t = u_{xx} \tag{11.1}$$

$$u(x,0) = u_0(x) \tag{11.2}$$

where $u_0(x)$ is a given function describing the initial condition. The same equation also describes diffusion of, e.g., a dye in water or perfume in air. The units of space and time have been chosen such that the thermal conductivity of the material is 1. For physical reasons u_0 is real, bounded and piecewise continuous, and hence $u_0 \in L^2(T)$. From the the heat equation it then follows that $u(\cdot,t) \in L^2(T)$ for all $t \geq 0$. Indeed:

$$\frac{d}{dt}\int_T |u(x,t)|^2\,dx = \int_T 2u(x,t)u_t(x,t)\,dx \tag{11.3}$$

$$= \int_T 2u(x,t)u_{xx}(x,t)\,dx \tag{11.4}$$

$$= -2\int_T |u_x(x,t)|^2\,dx \leq 0 \quad . \tag{11.5}$$

It follows that $u(x,t)$, for all fixed $t \geq 0$, can be expanded as a Fourier series:

$$u(x,t) = \sum_{n=-\infty}^{\infty} u_n(t)e^{inx} \tag{11.6}$$

with

$$u_n(t) = \frac{1}{2\pi}\int_0^{2\pi} u(x,t)e^{-inx}\,dx$$

The coefficients, which still depend on t, can be found by plugging (11.6) into the heat equation:

$$u_t(x,t) = \sum_{n=-\infty}^{\infty} \dot{u}_n\,(t)e^{inx} = \sum_{n=-\infty}^{\infty} -n^2 u_n(t)e^{inx} = u_{xx}(x,t)$$

By taking the inner product of both sides with e^{imt} we find that $u_n(t)$ must satisfy

$$\dot{u}_n = -n^2 u_n \tag{11.7}$$

$$u_n(0) = \frac{1}{2\pi}\int_0^{2\pi} u_0(x)e^{-inx}\,dx \tag{11.8}$$

Thus, the PDE is reduced to an infinite set of decoupled ODE. The solutions are

$$u_n(t) = u_n(0)e^{-n^2 t}$$

Summarizing we obtain that the solution of the heat equation on the unit circle with initial data u_0 is given by

$$u(x,t) = \sum_{n=-\infty}^{\infty} u_n(0)e^{-n^2 t}e^{inx} \tag{11.9}$$

The above arguments are a bit 'formal' if u_0 is not smooth, because the the RHS of the heat equation does not seem to be well-defined in that case. For comfort you can limit the discussion to smooth initial conditions, but this is not necessary, basically due to the first of the following three consequences of (11.9).

Properties the solution $u(x,t)$:

1. *Smoothing property:* For all $t > 0$, $u(x,t) \in C^\infty(T)$. This follows from

$$\sum_{n=-\infty}^{\infty} n^{2p}|u_n(0)e^{-n^2 t}|^2 \le \left[\max_n n^{2p}e^{-n^2 t}\right]\sum_m |u_m(0)|^2 < \infty$$

for all $p \ge 0$, and the Sobolev Embedding Theorem 4.3 (notes to be updated).

2. *Irreversibility:* Bad things may happen for $t < 0$, even for C^∞ 'final data'. E.g., $u_0(x) = \sum_n e^{-|n|}e^{inx} \in C^\infty(T)$, but there is no corresponding $u(x,-\epsilon)$ for any $\epsilon > 0$.

3. *Exponential decay and limiting behavior:* The temperature $u(x,t)$ converges exponentially fast to its average value as $t \to \infty$. This is expressed by the following estimate, valid for all $t > 0$:

$$\begin{aligned}
& \sup_{x\in T}\left|\sum_{n=-\infty}^{\infty} u_n(0)e^{-n^2t}e^{inx} - \frac{1}{2\pi}\int_0^{2\pi} u_0(x)\,dx\right| \\
\leq & \sum_{n\neq 0} |u_n| e^{-n^2 t} \\
\leq & \left[\sum_{n\neq 0} |u_n|^2\right]^{1/2} \left[2\sum_{n=1}^{\infty} e^{-2n^2t}\right]^{1/2} \\
\leq & \text{ constant} \times \|u_0\|_{L^2} e^{-t}
\end{aligned}$$

Note that the temperature averaged over the ring is independent of time. This can be seen by computing its derivative with respect to time and using the heat equation:

$$\frac{d}{dt}\int u(x,t)\,dx = \int u_t(x,t)\,dx = \int u_{xx}(x,t)\,dx = 0.$$

11.1.2 The heat equation on an interval

The only difference with the case of a ring is that one needs to specify boundary conditions, e.g., one can fix the temperatures at the end points of the interval.

Exercise 11.1 *Solve the heat equation*

$$\begin{aligned}
u_t &= u_{xx} \\
u(x,0) &= u_0(x), \quad 0 \leq x \leq L \\
u(0,t) &= u(L,t) = 0, \quad t > 0
\end{aligned}$$

Hint: Expand the initial data and the solution in a suitable basis of $L^2([0,L])$.

11.1.3 Temperature of the earth as a function of depth

If one is not interested in the daily fluctuations, a crude but reasonable model is the following. Assume that the surface temperature of the earth is a periodic function with the period equal to one year. It is then reasonable to expect that after many years, the temperature at a depth x below the surface will also be a periodic function of time with period one year. For simplicity, choose a time unit such that 1 year

$= 2\pi$. Then, for any fixed depth x, the temperature $u(x,t)$ can be expanded in a Fourier series:

$$u(x,t) = \sum_{n=-\infty}^{\infty} u_n(x) e^{int}$$

and, of course, has to satisfy the heat equation. This leads to the following ODE for the coefficients u_n as functions of x ($'$ denotes derivative with respect to x):

$$-inu_n = u_n'' \tag{11.10}$$

Of which the solutions are

$$u_n(x) = u_n(0)e^{\pm\sqrt{n}(1+i)x/\sqrt{2}} \quad \text{if } n \geq 0 \tag{11.11}$$

$$= u_n(0)e^{\pm\sqrt{|n|}(1-i)x/\sqrt{2}} \quad \text{if } n < 0 \tag{11.12}$$

The solutions with the + sign in the exponent, should be excluded because they blow up. These soltions can also be removed by imposing any reasonable boundary condition at infinite depth. Note that we are considering the heat equation on the half-line. The general solution is then:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(0)e^{-|n/2|^{1/2}x} e^{i(nt+|n/2|^{1/2}x)} + \sum_{n=-\infty}^{-1} u_n(0)e^{-|n/2|^{1/2}x} e^{i(nt-|n/2|^{1/2}x)} \tag{11.13}$$

For concreteness suppose that the surface temperatur is given by a simple harmonic function:

$$u(0,t) = A + B\sin t$$

Then, (11.13) becomes

$$u(x,t) = A + Be^{-x/\sqrt{2}} \sin(t - x/\sqrt{2})$$

The exponential damping factor in front of the sine function reduces the fluctuations of the earths temperature below the surface. The argument of the sine function indicates that there is a phase shift for the temperature fluctuations depending on the depth. At a depth x such that $x/\sqrt{2} = \pi$, the fluctuations are reduced by a factor of $e^{-\pi} \sim 0.04$, and are opposite in phase with the surface temperature. For realistic numerical values of the thermal conductivity of the soil this happens for a depth of about 13 feet. So, 13 feet below the surface maximum temperatures are reached in winter and minima in the summer! At the same time the difference between winter and summer temperature is reduced by a factor of about 25 as compared to the average daily temperature at the surface.

11.2 The wave equation: a vibrating string

The one-dimensional wave equation is

$$\frac{1}{c^2}u_{tt} = u_{xx} \tag{11.14}$$

where c is the speed of propagation of the wave (e.g., the speed of sound in a medium, or the speed of light in vacuum).

Note that

$$u(x,t) = \exp(i(kx - \omega t)) \tag{11.15}$$

is a solution of this equation whenever $\omega^2 = c^2k^2$. As this equation is second order in t one should expect that two initial conditions are needed to specify a unique solution. For concreteness, let us consider waves in a vibrating string of length L held fixed at the end points. The following is then a well-posed problem with a unique solution:

$$\frac{1}{c^2}u_{tt} = u_{xx} \tag{11.16}$$

$$u(0,t) = u(L,t) = 0 \tag{11.17}$$

$$u(x,0) = f(x) \tag{11.18}$$

$$u_t(x,0) = g(x) \tag{11.19}$$

where f and g are continuous functions satisfying the boundary conditions. By taking linear combinations of the solutions (11.15) we obtain the following with factorized x and t dependence and which real for real constants α, β, A, and B:

$$(\alpha \sin kx + \beta \cos kx)\,(A \sin \omega t + B \cos \omega) \tag{11.20}$$

To satisfy the boundary conditions one needs $\beta = 0$ and

$$k_n = \frac{n\pi}{L}, \quad \omega_n = \frac{cn\pi}{L}, \quad n = 1,2,3,\ldots$$

Due to the linearity of the equation all linear combinations of solutions of the form (11.20) will also be solutions. Therefore, we propose a general solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} \sin(k_n x)\,(A_n \sin \omega_n t + B_n \cos \omega_n t)$$

with real constants A_n, B_n.

Exercise 11.2 *Is the set* $\{\sqrt{2/L}\sin(n\pi x/L) \mid n = 1,2,\ldots\}$ *an o.n. basis for* $L^2([0,L])$*? Why (not)?*

Exercise 11.3 *Find sufficient conditions on the coefficients A_n and B_n such that $u(x,t)$ is continuous as a function of x for all times $t \in \mathbb{R}$.*

Requiring that the solition satisfy the initial conditions (11.18) and (11.19) determines A_n and B_n uniquely:

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L \sin(n\pi x/L) f(x)\,dx \\ \omega_n A_n &= \frac{2}{L}\int_0^L \sin(n\pi x/L) g(x)\,dx \end{aligned}$$

Note that for the wave equation soltions exist for both $t \geq 0$ and $t < 0$, and that there is no convergence to a stationary solution as $t \to \infty$, as we found for the heat equation.

Exercise 11.4 *Prove the following conservation law for the wave equation:*

$$0 = \frac{d}{dt}\int_0^L |u_t(x,t)|^2 + c^2 |u_x(x,t)|^2\,dx$$

11.3 The Schrödinger equation: a particle in a box

The Schrödinger equation for a particle in a one-dimensional box of size L is a PDE for a complex-valued function $u(x,t)$, $x \in [0,L]$, $t \in \mathbb{R}$, (called the *wavefunction* of the particle, and in the physics literature usually denoted by $\psi(x,t)$). A possible choice of boundary conditions is to require that the wavefunction vanishes at the boundary (*Dirichlet boundary conditions*). So, we are looking for solutions of

$$\begin{aligned} iu_t &= -u_{xx} && (11.21) \\ u(0,t) &= u(L,t) = 0, \quad \text{for all } t && (11.22) \\ u(x,0) &= \psi_0(x) && (11.23) \end{aligned}$$

There are two important conservation laws for this equation. The first one is that the L^2-norm of any solution is independent of time:

$$\begin{aligned} \frac{d}{dt}\int_0^L |u(x,t)|^2\,dx &= \int_0^L \overline{u_t}\,u\,dx + \int_0^L \overline{u}\,u_t\,dx \\ &= \int_0^L \overline{iu_{xx}}\,u\,dx + \int_0^L i\overline{u}\,u_{xx}\,dx \\ &= i\int_0^L \overline{u_x}\,u_x\,dx - i\int_0^L \overline{u_x}\,u_x\,dx = 0 \end{aligned}$$

This property implies that if $\psi_0 \in L^2([0,L])$, $u(\cdot,t) \in L^2([0,L])$, for all t. This property is often called *unitarity*, for reasons that will become clear later. Note that the boundary conditions are essential for this unitarity property.

Exercise 11.5 *Prove unitarity for the Schrödinger equation with Neumann boundary conditions, i.e., $u'(0,t) = u'(L,t) = 0$.*

The physical interpretation of $|u(x,t)|^2$ is that it is proportional to the probability distribution of the position of the particle at time t. I.e.,

$$\text{prob}\begin{pmatrix}\text{at time } t \text{ the particle is observed}\\ \text{to be in the interval } [a,b]\end{pmatrix} = \frac{\int_a^b |u(x,t)|^2\,dx}{\int_0^L |u(x,t)|^2\,dx}$$

for $0 \leq a \leq b \leq L$. In this interpretation unitarity means conservation of probability, or conservation of the normalization of the probability distribution.

A second important conservation law is conservation of energy given by

$$\frac{d}{dt}\int_0^L |u_x|^2\,dx = 0 \tag{11.24}$$

Exercise 11.6 *Verify (11.24).*

Using Fourier series one can again find the general solutions of (11.21):

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(k_n x) e^{-i\omega_n t}$$

where

$$k_n = \frac{\pi n}{L}, \quad \omega = \left(\frac{\pi n}{L}\right)^2$$

Stationary solutions exhibit the phenomenon called *quantization of energy*. Here, stationary can be defined to mean that $|u(x,t)|^2$ is independent of time, which holds if only one term in (11.3) is non-vanishing. Then, the energy integral equals one of the discrete values ω_n:

$$\frac{\int_0^L |u_x|^2\,dx}{\int_0^L |u|^2\,dx} = \left(\frac{\pi n}{L}\right)^2 = \omega_n$$

The stationary solutions are also the solutions of the following eigenvalue problem:

$$-\frac{d^2}{dx^2}u(x) = \lambda u(x) \tag{11.25}$$

$$u(0) = u(L) = 0 \tag{11.26}$$

$$u \in L^2([0,L]) \tag{11.27}$$

More generally, in quantum mechanics stationary solutions are found by solving an (infinite-dimensional) eigenvalue problem of this kind. We will study those in considerable detail later.

11.4 A dispersive wave equation: the linearized KdV equation

In 1895 Korteweg and de Vries introduced a non-linear PDE to describe water waves in shallow channels:

$$u_t = uu_x + u_{xxx} \tag{11.28}$$

There are two remarkable facts about this equation: 1) it has solitary wave solutions (solitons) and 2) in spite of the fact that it is non-linear, it can be solved exactly (by the so-called inverse scattering method poineered by Gardner, Greene, Kruskal, and Miura 1967). This goes beyond simple Fourier series and we do not intend to discuss the inverse scattering method in this course. There is one property, however, that can be illustrated by considering the linearized version of the KdV equation, namely that this equation is *dispersive* (see below). As long as u is small enough, the term uu_x is not expected to influence the solutions very much, and it can omitted in a first approximation. So, let us consider

$$\begin{aligned} u_t &= u_{xxx} && (11.29)\\ u(x,0) &= u_0(x) && (11.30)\\ u(\cdot,t) &\in L^2(T), \quad \text{for all } t && (11.31) \end{aligned}$$

The solution is

$$u(x,t) = \sum_{n=-\infty}^{\infty} a_n e^{-in^3t} e^{inx} = \sum_{n=-\infty}^{\infty} a_n e^{in(x-n^2t)}$$

Notice that the speed of wave propagation depends on n, i.e, on the wavelength. As components in the wave with different wavelengths travel at different speeds, a wavefront is generally spreading out. Hence the name *dispersive* waves. This has to be compared the solutions of

$$\begin{aligned} u_{tt} &= u_{xx} && (11.32)\\ u(x,0) &= u_0(x) && (11.33)\\ u(\cdot,t) &\in L^2(T), \quad \text{for all } t && (11.34) \end{aligned}$$

of which the solution is

$$u(x,t) = \sum_{n=-\infty}^{\infty} a_n^+ e^{in(x+t)} + a_n^- e^{in(x-t)}$$

which is non-dispersive.

12 More applications of Fourier Series

12.1 The isoperimetric problem

Of all smooth (or piecewise smooth) closed curves in the plane of a given length, say 1, the ones that maximize the area they enclose are the circles. This result is sometimes stated as an inequality: for any smooth closed curve of length L and enclosing an area A one has

$$\frac{A}{L^2} \leq 4\pi \tag{12.1}$$

with equality iff the curve is a circle. This is called (the) an isoperimetric inequality.

A proof using Fourier series is given in the book by Dym and McKean, *Fourier Series and Integrals*. A sketch of the proof is in Naylor and Seel, it Linear Operator Theory in Engineering and Science, Section 5.18, Ex. 6 (pp 329–330).

12.2 Ergodicity of arithmetic sequences

Our last application is an ergodic theorem for one of the simplest dynamical systems one can imagine. Let $\gamma \in \mathbb{R}$ and consider the following map on the circle:

$$F : T \to T, \theta \mapsto \theta + 2\pi\gamma \tag{12.2}$$

This map is called the *circle map* or *rotation map*. For every $\theta_0 \in T$ the iterated application of F generates an infinite sequence $(\theta_n = F^n(\theta_0))_{n=0}^{\infty}$ in T.

For any $f \in C(T)$ we can define two averages, a 'time' average, $\langle f \rangle_t$, and a 'phase-space' average, $\langle f \rangle_{ph}$, by

$$\begin{aligned} \langle f \rangle_t &:= \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} f(\theta_n) \\ \langle f \rangle_{ph} &:= \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta \end{aligned}$$

Theorem 12.1 (Weyl, 1916) *If γ is irrational, then*

$$\langle f \rangle_t = \langle f \rangle_{ph} \tag{12.3}$$

for all $f \in C(T)$ and all $\theta_0 \in T$.

Proof: First, we show the equality (12.3) for the functions $e_m(\theta) = e^{im\theta}$, $m \in \mathbb{Z}$. The case $m = 0$ is trivial: both averages are equal to 1. So, suppose, $m \neq 0$. Then

$\langle e_m \rangle_{ph} = 0$. And the time averages can be easily computed as follows:

$$\begin{aligned}
\langle e_m \rangle_t &= \lim_N \frac{1}{N+1} \sum_{n=0}^{N} e^{im(\theta_0 + 2\pi n\gamma)} \\
&= \lim_N \frac{1}{N+1} e^{im\theta_0} \sum_{n=0}^{N} [e^{2\pi i m\gamma)}]^n \\
&= \lim_N \frac{1}{N+1} e^{im\theta_0} \frac{1 - [e^{2\pi i m\gamma}]^{N+1}}{1 - e^{2\pi i m\gamma}} \\
&= 0
\end{aligned}$$

where we used the fact that $e^{2\pi i m\gamma} \neq 1$ for *irrational* γ.

Both averages are linear in f. So (12.3) also holds for arbitrary linear combinations of the functions e_m, i.e., it holds for the trigonometric polynomials.

Next, we use the fact that the trigonometric polynomials are dense in $C(T)$ with the uniform norm. I.e., for $f \in C(T)$ and any $\epsilon > 0$ there is a trigonometric polynomial P such that $\|P - f\|_{\sup} \leq \epsilon$. Pick such a P. Then

$$\left| \frac{1}{N+1} \sum_{n=0}^{N} f(\theta_n) - \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta \right| \leq 2\epsilon + \left| \frac{1}{N+1} \sum_{n=0}^{N} P(\theta_n) - \frac{1}{2\pi} \int_0^{2\pi} P(\theta)\, d\theta \right|$$

By taking $\limsup_N$ we obtain

$$|\langle f \rangle_t - \langle f \rangle_{ph}| \leq 2\epsilon$$

As ϵ is arbitrary this proves (12.3) for all $f \in C(T)$ and $\theta_0 \in T$. ■

Note that for rational γ (12.3) does not hold for most f.

Corollary 12.2 *For any interval* $[\theta_1, \theta_2] \subset T$*, and irrational* γ*, one has*

$$\lim_{N\to\infty} \frac{1}{N+1} \#\{n \mid 0 \leq n \leq N, \theta_n \in [\theta_1, \theta_2]\} = \frac{|\theta_1 - \theta_2|}{2\pi} \tag{12.4}$$

The relation (12.4) expresses the property that the points of the sequence θ_n are uniformily distributed on the circle.

Exercise 12.1 *As a math writing exercise, write a proof of Corollary 12.2.*