16 The spectrum of bounded linear operators

16.1 Motivation

We have used Fourier series to solve, e.g., the heat equation on an interval $[0, L]$, with given boundary conditions and inital data. What would it take to express the solution of

$$\begin{aligned} u_t = u_x x + qu & \quad 0 \leq x \leq L, t \geq 0 \\ u(0,t) = u(L,t) = 0 & \quad t \geq 0 \\ u(x,0) = f(x) & \end{aligned}$$

for given functions f and q, as a series expansion similar to a Fourier series. This equation describes the temperature of a heat conducting bar with a non-uniform heat loss term given by $-q(x)u(x,t)$. If we try separation of variables and assume

$$u(x,t) = \sum_n a_n(t) u_n(x)$$

where $\{u_n\}$ is an o.n. basis of $L^2([0, L])$, and where the $a_n(t)$ are solutions of $\dot{a_n} = \lambda_n a_n$, with constants λ_n, we find that u_n should satisfy

$$\frac{d^2}{x^2} u_n(x) + q(x) u_n(x) = \lambda_n u_n$$

I.e., the u_n should be eigenvectors of the linear operator D defined by

$$Df(x) = \frac{d^2}{x^2} f(x) + q(x) f(x)$$

Diagonalization problems of linear maps on infinite-dimensional spaces arise in many other ways. The mathematics of this so-called spectral problem is the main topic of this course for the rest of the quarter. This is again a generalization of something we know well in the finite-dimensional case: diagonalization of matrices. First, we briefly review the finite-dimensional situation.

16.2 Diagonalization of matrices

We consider an $n \times n$ matrix A with complex entries as a (bounded) linear map $\mathbb{C}^n \to \mathbb{C}^n$. A complex number λ is called an *eigenvalue* of A if there is a non-zero vector $u \in \mathbb{C}^n$ such that

$$Au = \lambda u \tag{16.1}$$

A non-zero vector $u \in \mathbb{C}^n$ such that (16.1) holds for $\lambda \in \mathbb{C}$, is called an *eigenvector* of A (belonging to the eigenvalue λ).

One says that A is *diagonalizable* if there is a basis $\{u_1, \ldots, u_n\}$ of $\mathbb{C}^n$, consisting of eigenvectors, i.e.,

$$Au_k = \lambda u_k, \quad \lambda_k \in \mathbb{C}, k = 1, \ldots, n$$

The set of eigenvalues of A is called the *spectrum* of A, often denoted by $\sigma(A)$.

In this course we will stick with the Hilbert space setting, where the interesting bases are o.n. bases.

When does an $n \times n$ matrix have an o.n. basis of eigenvectors? If there is an o.n. basis $\{u_1, \ldots, u_n\}$ such that $Au_k = \lambda_k u_k$, there exists a unitary U defined by

$$Ue_k = u_k$$

where $\{e_k\}$ is the standard o.n. basis of $\mathbb{C}^n$. Clearly one then also has $U^* u_k = U^{-1} u_k = e_k$, $k = 1, \ldots n$, and therefore

$$U^* A U e_k = \lambda_k e_k$$

which means that U^*AU is a diagonal matrix with the eigenvalues of A on the diagonal:

$$U^*AU = D := \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Or, equivalently, $A = UDU^*$, with U unitary and D diagonal.

When can a matrix A that can be written in the form UDU^*? If $A = UDU^*$, we also have $A^* = U\overline{D}U^*$, and

$$A^*A = U\overline{D}U^*UDU^* = U\overline{D}DU^* = UD\overline{D}U^* = UDU^*U\overline{D}U^* = AA^*$$

In other words $A^*A = AA^*$, i.e., A commutes with its hermitian conjugate. Matrices with this property are called *normal* matrices (or, more generally, normal operators). So, we found that matrices that have an o.n. basis of eigenvectors are normal. The converse also holds.

Theorem 16.1 *For any normal $n \times n$ matrix there is an o.n. basis of eigenvectors.*

This is a standard theorem of linear algebra.

Special cases of normal matrices are the *hermitian* matrices (satisfying $A^* = A$), and the *unitary* matrices (satisfying $U^*U = \mathbb{1} = UU^*$).

Any normal martrix N can be written as the product of a unitary matrix V and a hermitian matrix A. A can be chosen to be non-negative definite. This follows directly from the diagonal form of N: if $N = UDU^*$, with eigenvalues $\lambda_k = e^{i\varphi_k}|\lambda_k,$

we can write $D = \Phi|D|$, where Φ is a diagonal matrix with $e^{i\varphi_k}$ on the diagonal and $|D|$ is diagonal with $|\lambda_k|$ on the diagonal. Then,

$$N = UDU^* = (U\Phi U^*)(U|D|U^*) = VA \tag{16.2}$$

with $V = (U\Phi U^*)$, as a product of unitaries, is unitary, and $A = (U|D|U^*)$ is non-negative definite. It is easy to check that $VA = AV$. (16.2) is called the *polar decomposition* of N.

The converse is also true: if $N = VA$, with V unitary, and A hermitian, and such that $VA = AV$, then N is normal.

The eigenvalues are the roots of the polynomial of degree n given by $P(x) = \det(A - x\mathbb{1})$. And if $P(\lambda) = \det(A - \lambda\mathbb{1}) = 0$ we have that $\ker(A - \lambda\mathbb{1} \neq \{0\}$, i.e., there is a non-zero vector u such that $Au = \lambda u$. It follows that all matrices have at least one eigenvalue, and that to each eigenvalue there corresponds a non-zero subspace of eigenvectors.

The following countereample shows that not all matrices have a basis of eigenvectors:

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

λ is the only eigenvalue of A and the eigenvectors belonging to λ are multiples of $u = (1, 0)$, which does not contain a basis of $\mathbb{C}^2$.

16.3 The spectrum: definition

It turns out that it is useful to define the spectrum of a linear operator on an infinite-dimensional space a bit more generally than just the set of eigenvalues. As we will see in an example below, in inifinite dimensions, a bounded linear operator does not necessarily have any eigenvalues (and eigenvectors), even if it is self-adjoint ($A^* = A$).

As before, let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote the Banach space of all bounded linear transformations of $\mathcal{H}$.

Definition 16.2 (Resolvent Set, Spectrum) *For $A \in \mathcal{B}(\mathcal{H})$, the resolvent set of A, $\rho(A)$, is the set of all complex numbers λ such that*

$$A - \lambda\mathbb{1}$$

is one-to-one and onto and (hence[2]) has a bounded inverse. The spectrum of A, $\sigma(A)$, is the complement of the resolvent set: $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

[2]Due to the open mapping theorem (see, e.g., M. Reed and B. Simon, *Modern Methods of Mathematical Physics*, Vol I, Theorem III.11.)

Example: If λ is an eigenvalue of A in the sense that there is a non-zero vector $u \in \mathcal{H}$ such that $Au = \lambda u$, then $\ker(A - \lambda \mathbb{1}) \neq \{0\}$, and hence $A - \lambda \mathbb{1}$ is not one-to-one. It follows that $\lambda \in \sigma(A)$.

This example, however, does not cover all the possible reasons for which a complex number λ may belong to the spectrum:

- $A - \lambda \mathbb{1}$ is not one-to-one. In this case λ is called an *eigenvalue*. The set of all eigenvalues of A is called the *point spectrum* of A.

- $A - \lambda \mathbb{1}$ is one-to-one, but, although $\operatorname{ran}(A - \lambda \mathbb{1})$ is dense in $\mathcal{H}$, $A - \lambda \mathbb{1}$ is not onto, i.e., $A - \lambda \mathbb{1}$ does not have a *bounded* inverse. In this case λ is said to belong to the *continuous spectrum* of A.

- It is also possible that $A - \lambda \mathbb{1}$ is one-to-one, but $\operatorname{ran}(A - \lambda \mathbb{1})$ is not dense in $\mathcal{H}$, in which case we say that λ belongs to the *residual spectrum* of A.

Here is an example of an operator without any eigenvalues. This is to be contrasted with the finite-dimensional situation where there always is at least one eigenvalue.

Example: Let $\mathcal{H} = L^2([0,1])$, and define $X \in \mathcal{B}(\mathcal{H})$ by

$$Xf(x) = xf(x)$$

It is easy to see that X is bounded and that $\|X\| = 1$. For any $\lambda \in \mathbb{C}$, suppose there is a function f such that $Xf = \lambda f$. Then $f(x) = 0$, for all $x \neq \lambda$. Hence $f = 0 \in L^2$. So, X has no eigenvalues.

On the other hand, $\operatorname{ran}(X - \lambda \mathbb{1})$ is dense for all $\lambda \in \mathbb{C}$. In fact for $\lambda \in \mathbb{C} \setminus [0,1]$, $\operatorname{ran}(X - \lambda \mathbb{1}) = L^2$: as $x \mapsto (x - \lambda)^{-1} \in L^2([0,1])$, is bounded and continuous we have that for any $f \in L^2([0,1])$, $(x-\lambda)^{-1} f(x) \in L^2([0,1])$, and $(X - \lambda \mathbb{1})(x - \lambda)^{-1} f(x) = f(x)$. As f is arbitrary, this proves that $\operatorname{ran}(X - \lambda \mathbb{1}) = L^2([0,1])$.

For $\lambda \in [0,1]$, the non-zero constant functions $f(x) = c$ are not in the range of $X - \lambda \mathbb{1}$, because $c(x - \lambda)^{-1} \notin L^2([0,1])$. However $\operatorname{ran}(X - \lambda \mathbb{1})$ is dense also in this case:

$$\lim_n (X - \lambda \mathbb{1}) \operatorname{Ind}[|x - \lambda| \geq 1/n] \frac{f}{x - \lambda} = f, \quad f \in L^2([0,1])$$

where the limit is in the sense of L^2.

We have proved that $\sigma(X) = [0,1]$, and that all $\lambda \in [0,1]$ belong to the continuous spectrum of X.

Note that if a Kronecker delta $\delta_{x,\lambda}$, $0 \leq \lambda \leq 1$, (or a "delta function") would be a non-zero element of the space on which X acts, it would be an eigenvector with eigenvalue λ. In some sense it is still true that the elements of the spectrum are similar to eigenvalues.

16.4 Elementary properties of the spectrum

Recall that the resolvent set $\rho(A)$ of a linear operator A is the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda\mathbb{1}$ has an everywhere defined, bounded inverse. It is conventional to denote the inverse of $\lambda\mathbb{1} - A$ by R_λ. R_λ is called the *resolvent* (or *resolvent operator*) of A (at λ). The following proposition and its proof show that R_λ can be considered as an (operator-valued) analytic function defined on the open domain $\rho(A)$.

Proposition 16.3 *For any $A \in \mathcal{B}(\mathcal{H}), \rho(A)$ is an open subset of $\mathbb{C}$.*

Proof: We have to show that for any $\lambda_0 \in \rho(A)$, there is an open disk in the complex plane with center λ_0, and entirely contained in $\rho(A)$. We start from

$$\lambda\mathbb{1} - A = (\lambda_0\mathbb{1} - A)\left[\mathbb{1} - (\lambda_0 - \lambda)(\lambda_0\mathbb{1} - A)^{-1}\right]$$

If $|\lambda_0 - \lambda| < \|(\lambda_0\mathbb{1} - A)^{-1}\|^{-1}$ we can invert the RHS (and hence the LHS) by the Neumann series . ■

The Neumann series also shows that

$$\sigma(A) \subset \{z \in \mathbb{C} \mid |z| \leq \|A\|\}$$

The *spectral radius* of A, $r(A)$, is defined as the radius of the smallest disk with center 0 that contains $\sigma(A)$:

$$r(A) := \sup\{|\lambda| \mid \lambda \in \sigma(A)\}$$

One can show that for self-adjoint operators A, $r(A) = \|A\|$. But, in general, one only has $r(A) \leq \|A\|$. E.g., consider $\mathcal{H} = \mathbb{C}^2$ and

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Then $r(A) = 0$, and $\|A\| = 1$.

Proposition 16.4 *For any $A \in \mathcal{B}(\mathcal{H})$, $\sigma(A)$ is not empty.*

Proof: For sufficiently large $|\lambda|$, i.e., $|\lambda| > \|A\|$, the resolvent $R_\lambda = (\lambda\mathbb{1} - A)^{-1}$ is bounded and given by a norm-convergent series:

$$R_\lambda = \frac{1}{\lambda}\left[\sum_{n=0}^{\infty}\left(\frac{A}{\lambda}\right)^n\right]$$

It follows that for any $x, y \in \mathcal{H}$, the function f defined by

$$f(\lambda) := \langle x, R_\lambda y\rangle \tag{16.3}$$

is given by a convergent series expansion as well, and is analytic and uniformily bounded for $|\lambda| > \|A\|$. Its domain of analyticity is $\rho(A)$. Indeed, f is analytic because it is given by a power series on $\rho(A)$ (previous proposition). (Note also that R_λ cannot be analytic on a domain containing points of the spectrum, because then, as in the proof of the previous proposition, $\lambda\mathbb{1} - A$ would be invertible for those λ.) So, if $\sigma(A)$ were empty, f would be a bounded entire function. By Liouville's Theorem, this implies that f is a constant function, and as $\lim_{\lambda\to\infty} f(\lambda) = 0$, this means $f = 0$. $f = 0$ for all choices if $x, y \in H$, in (16.3), would imply $R_\lambda = 0$ for all λ, which is impossible. Therefore, we must conclude that $\sigma(A) \neq \emptyset$ (and thus alow for singularites in the functions f). ∎

The following lemma gives an important relation between an operator and its adjoint.

Lemma 16.5 *For any $A \in \mathcal{B}(\mathcal{H})$ the following relations hold:*

$$(\ker A)^\perp = \overline{\mathrm{ran} A^*} \tag{16.4}$$

$$(\mathrm{ran} A)^\perp = \ker A^* \tag{16.5}$$

Proof: If $x \in \mathrm{ran} A^*$, there exists $y \in \mathcal{H}$ such that $x = A^* y$. For any $z \in \ker A$, we then have

$$\langle x, z\rangle = \langle A^* y, z\rangle = \langle y, Az\rangle = 0$$

This proves that $\mathrm{ran} A^* \subset (\ker A)^\perp$. As $(\ker A)^\perp$ is closed, this implies $\overline{\mathrm{ran} A^*} \subset (\ker A)^\perp$.

On the other hand, if $x \in (\mathrm{ran} A^*)^\perp$, we have, for all $y \in \mathcal{H}$,

$$0 = \langle A^* y, x\rangle = \langle y, Ax\rangle$$

and therefore, $Ax = 0$. This means $(\mathrm{ran} A^*)^\perp \subset \ker A$. By taking the $\perp$ complement of both sides of this relation we get $(\ker A)^\perp \subset (\mathrm{ran} A^*)^{\perp\perp} = \overline{\mathrm{ran} A^*}$. This concludes the proof of the first relation.

The second relation follows from the first by applying it to A^*, instead of A, and by using $A^{**} = A$, and taking orthogonal complements. ∎

Proposition 16.6 *For any $A \in \mathcal{B}(\mathcal{H})$, if λ is in the residual spectrum of A, then $\overline{\lambda}$ is an eigenvalue of A^*.*

Proof: If λ belongs to the residual spectrum of A, we have that $\mathrm{ran}(\lambda\mathbb{1} - A)$ is not dense in $\mathcal{H}$. This means that there exists $0 \neq x \in (\mathrm{ran}(\lambda\mathbb{1} - A))^\perp = \ker(\lambda\mathbb{1} - A)^* = \ker(\overline{\lambda}\mathbb{1} - A^*)$, where we have used the previous lemma. ∎

Exercise 16.1 *Prove that $\rho(A^*) = \overline{\rho(A)}$.*

Exercise 16.2 *If λ is an eigenvalue of A, $\overline{\lambda}$ is in the spectrum of A^*. What can you say about the type of spectrum $\overline{\lambda}$ belongs to?*

Lemma 16.7 *If $A = A^* \in \mathcal{B}(\mathcal{H})$, we have that $\sigma(A) \subset [-\|A\|, \|A\|] \subset \mathbb{R}$.*

Proof: We already know that $r(A) \leq \|A\|$. So we only have to prove that the spectrum of a self-adjoint A consists only of real numbers.

Write $\lambda = a + ib$, with $a, b \in \mathbb{R}$. Then, for all $x \in \mathcal{H}$

$$\begin{aligned}
\|(A - \lambda 1\!\!1)x\|^2 &= \langle (A - \lambda 1\!\!1)x, (A - \lambda 1\!\!1)x \rangle \\
&= \langle (A - a1\!\!1)x, (A - a1\!\!1)x \rangle + \langle (-ib)x, (-ib)x \rangle + \langle Ax, (-ib)x \rangle + \langle (-ib)x, Ax \rangle \\
&= \|(A - a1\!\!1)x\|^2 + b^2\|x\|^2 \geq b^2\|x\|^2
\end{aligned}$$

Suppose $\lambda \in \sigma(A)$ and $b \neq 0$. Then, λ cannot be an eigenvalue because the inequality shows that $\ker(A - \lambda) = 0$. Also, as $b \neq 0$, the inequality above implies that $(A - \lambda 1\!\!1)$ has a bounded inverse on its range (see Exercise 16.3 below). It is then easy to check that $\mathrm{ran}(A - \lambda 1\!\!1)$ is closed. If $\mathrm{ran}(A - \lambda 1\!\!1) \neq \mathcal{H}$, λ would therefore be in the residual spectrum of A, and by Proposition 16.6, $\overline{\lambda} = a - ib$, would then be an eigenvalue of $A^* = A$. But this would contradict the inequality given above, and we conclude that $\lambda \in \rho(A)$ if $b \neq 0$. ■

Exercise 16.3 *Let X and Y be normed linear spaces and $T : X \to Y$ a linear map.*
a) *Suppose there exists a constant $c > 0$ such that*

$$\|Tx\| \geq c|x\| \quad . \tag{16.6}$$

Prove that T has a bounded inverse defined on $\mathrm{ran}(T)$, with $\|T^{-1}\| \leq c^{-1}$.
b) *Prove the converse of part a): if T has a bounded inverse defined on its range, then there exist a constant $c > 0$ such that (16.6).*

Corollary 16.8 *If A is self-adjoint the residual spectrum of A is empty.*

Proof: As the point spectrum and the residual spectrum are subsets of $\mathbb{R}$ (previous lemma) with empty intersection, this follows immediately from Proposition 16.6. ■

Exercise 16.4 *Prove that the spectrum of an orthogonal projection P is either $\{0\}$, in which case $P = 0$, or $\{1\}$, in which case $P = 1\!\!1$, or else $\{0, 1\}$.*

The following set of exercises provide excellent examples of the different kinds of spectrum. The shift operators appearing in these exercises can be used, e.g., to define difference equations, which, in turn, show up in numerical approximation schemes for ODE and PDE. See Exercise 16.9 for an example.

Exercise 16.5 *Let $\mathcal{H} = l^2(\mathbb{Z})$. The right shift S on $\mathcal{H}$ is defined by*

$$S(x)_k = x_{k-1}, \qquad \text{for all } k \in \mathbb{Z} \tag{16.7}$$

where $x = (x_k)_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$.

a) *Prove that the point spectrum of S is empty, i.e., S has no eigenvalues, or, equivalently, $\lambda\mathbb{1} - S$ is one-to-one for all $\lambda \in \mathbb{C}$.*

b) *Prove that* $\operatorname{ran}(\lambda\mathbb{1} - S) = l^2(\mathbb{Z})$, *for all $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. In other words, prove that the spectrum of S lies entirely in the unit disk.*

c) *Prove that* $\operatorname{ran}(\lambda\mathbb{1} - S) = l^2(\mathbb{Z})$, *for all $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. In other words, in combination with the previous part, prove that the spectrum of S lies entirely in the unit circle.*

d) *Prove that all $\lambda \in \mathbb{C}, |\lambda| = 1$, belong to the continuous spectrum of S.*

It is interesting to compare the results of the previous exercise to the following two, which concern the spectrum of similar shift operators on $l^2(\mathbb{N})$.

Exercise 16.6 *Define the right shift R on $l^2 := l^2(\{0, 1, 2, 3, \ldots\})$ by*

$$R(x) = (0, x_0, x_2, \ldots) \text{ for all } x = (x_0, x_1, x_2, \ldots) \in l^2 \quad . \tag{16.8}$$

a) *Prove that* $\operatorname{ran}(\lambda\mathbb{1} - R) = l^2(\mathbb{Z})$, *for all $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. In other words, prove that the spectrum of R lies entirely in the unit disk.*

b) *Prove that all $\lambda \in \mathbb{C}, |\lambda| = 1$, belong to the continuous spectrum of R.*

c) *Prove that* $\operatorname{ran}(\lambda\mathbb{1} - R)$ *is not dense in l^2, for all $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. In other words, prove that the interior of the unit disk is residual spectrum of R.*

Exercise 16.7 *Define the left shift L on $l^2 := l^2(\{0, 1, 2, 3, \ldots\})$ by*

$$L(x) = (x_1, x_2, \ldots) \text{ for all } x = (x_0, x_1, x_2, \ldots) \in l^2 \quad . \tag{16.9}$$

Prove the following:

a) *The resolvent set of L consist of all $\lambda \in \mathbb{C}, |\lambda| > 1$.*

b) *The continuous spectrum of L is the unit circle.*

c) *The point spectrum of L is given by the interior of the unit disc.*

d) *The residual spectrum of L is empty.*

Exercise 16.8 *Compute the adjoints S^*, R^*, and L^* of the operators S, R, and L defined in the previous exercises.*

Exercise 16.9 *Define an operator Δ on $l^2(\mathbb{Z})$ by*

$$(\Delta x)_k = x_{k-1} - 2x_k + x_{k+1}, \text{ for all } x = (x_k)_{k\in\mathbb{Z}} \in l^2(\mathbb{Z}) \ . \tag{16.10}$$

Prove that the spectrum of Δ is entirely continuous, real, and given by the interval $[-4, 0]$. (Hint: Note that $\Delta = S + S^* - 2\mathbb{1}$.)