# 17 The spectral theorem for compact self-adjoint operators

Let $X$ and $Y$ be Banach spaces (e.g., $X = Y = \mathcal{H}$, a Hilbert space).

**Definition 17.1** *A linear operator $T : X \to Y$ is called compact if for every bounded set $B \subset X$, the set $T(B)$ is pre-compact in $Y$.*

An equivalent formulation is: $T$ is compact if the image of any bounded sequence $x_n \in X$, has a convergent subsequence $Tx_{n_k} \to y \in Y$.

**Example:**
An orthogonal projection on a Hilbert space is compact iff its range is finite-dimensional.

**Proposition 17.2** *Let $X, Y$, and $Z$ be Banach spaces. Compact linear maps have the following properties:*

1. *If $C : X \to Y$ is compact then $C$ is bounded.*
2. *If $C_1, C_2 : X \to Y$ are compact then $C_1 + C_2$ is compact.*
3. *If $C_n \to C$ in norm, and $C_n : X \to Y$ is compact for all $n$, then $C$ is compact.*
4. *If $A : X \to Y$ and $B : Z \to X$ are bounded linear maps, and $C : Y \to Z$ is compact, then $CA$ and $BC$ are compact.*

It follows from this proposition that the compact linear operators $X \to Y$ form a Banach subspace of $\mathcal{B}(X, Y)$. And that the compact operators $X \to X$ are a two-sided ideal of $\mathcal{B}(X)$.

**Exercise 17.1** *Prove Proposition 17.2.*

We now focus on the problem of diagonalizing compact self-adjoint operators on a Hilbert space $\mathcal{H}$. We will give two versions of the spectral theorem for compact self-adjoint operators, one for general Hilbert spaces (Theorem 17.4), and one for separable Hilbert spaces (Theorem 17.3). This is merely for convenience of the user. In fact, the two versions have the same proof (see p. 136)

**Theorem 17.3 (Spectral Theorem, version 1)** *Let $\mathcal{H}$ be a separable Hilbert space and suppose $A = A^* \in \mathcal{B}(\mathcal{H})$ is compact. Then there is an o.n. basis $\{e_n\}$ of eigenvectors $\mathcal{H}$ with real eigenvalues $\lambda_n$, such that*

$$A = \sum_n \lambda_n \langle e_n, \cdot \rangle e_n \tag{17.1}$$

*The multiplicity of each non-zero eigenvalue is finite, and, if $\mathcal{H}$ is infinite-dimensional,* $\lim_n \lambda_n = 0$.

Note that if $\mathcal{H}$ is not separable, and $C$ is compact and self-adjoint, we necessarily have that $\mathcal{H}_0 := (\ker C)^\perp$ is a separable sub-Hilbert space of $\mathcal{H}$ (why ?), and $\mathcal{H}_0$ is an invariant subspace for $C$. We can therefore use the previous theorem to diagonalize $C$ restricted to $\mathcal{H}_0$, and, as $C = 0$ on $\mathcal{H}_0^\perp$ this also diagonalizes $C$ on the whole of $\mathcal{H}$. So, the condition of separability in the theorem is not really a restriction and the theorem can also be formulated as follows:

**Theorem 17.4 (Spectral Theorem, version 2)** *Let $\mathcal{H}$ be a (not necessarily separable) Hilbert space and suppose $A = A^* \in \mathcal{B}(\mathcal{H})$ is compact. Then $A$ has at most a countable number of non-zero eigenvalues $\lambda_n \in \mathbb{R}$, with a corresponding orthonormal set $\{e_n\}$ of eigenvectors, such that*

$$A = \sum_n \lambda_n \langle e_n, \cdot \rangle e_n \tag{17.2}$$

*The multiplicity of each non-zero eigenvalue is finite, and 0 is the only possible accumulation point of $\{\lambda_n\}$ (and 0 is an accumulation point if the set of non-zero eigenvalues is infinite).*

The relation (17.1) means that the sequence of operators defined by the partial sums,

$$S_N x = \sum_{n \le N} \lambda_n \langle e_n, x \rangle e_n, \qquad \text{for all } x \in \mathcal{H}$$

converges to $A$ in the operator norm.

It is easy to see why the non-zero eignvalues of a compact self-adjoint $A$ have finite multiplicity, and that if there are an infinite number of those, one must have that $\lim_n \lambda_n = 0$. Indeed if $\lambda \neq 0$ would have infinite multiplicity, the corresponding o.n. eigenvectors $e_n$ would form a bounded sequence, but $Ae_n = \lambda e_n$ does not have a convergent subsequence contradicting the assumption that $A$ is compact. If $A$ has an infinite number of non-zero eigenvalues $\lambda_n$, then, as the eigenvalues are all bounded by $\|A\|$, there are convergent subsequences $\lambda_{n_k} \to \lambda$. If $\lambda \neq 0$, the sequence of eigenvectors $\lambda_n^{-1} e_n$, is bounded, but again, $A\lambda_n^{-1} e_n = e_n$ does not have a convergent subsequence, which is impossible since $A$ is compact.

Before we embark on the proof of this theorem we will first prove some lemmas that we will need in the proof, but that have other uses as well.

After the proof of the theorem we will address the question how one can verify that a bounded linear operator is compact. We will also see how the theorem can be used to diagonalize certain *unbounded* (hence non-compact) linear operators.

**Lemma 17.5** *If $A = A^* \in \mathcal{B}(\mathcal{H})$, we have that*

$$\|A\| = \sup_{x, \|x\|=1} |\langle x, Ax \rangle|$$

**Proof:** One inequality is obvious:

$$|\langle x, Ax\rangle| \leq \|Ax\|\,\|x\| \leq \|A\|\,\|x\|^2$$

and therefore,

$$a := \sup_{x,\|x\|=1} |\langle x, Ax\rangle| \leq \|A\|$$

To prove the second inequality we use Lemma 17.5 to express $\|A\|$ as follows:

$$\|A\| = \sup_{x,\|x\|=1} \|Ax\| = \sup_{x,y;\|x\|=\|y\|=1} |\langle y, Ax\rangle|$$

and then polarization to rewrite the RHS as

$$\begin{aligned}&\langle y, Ax\rangle\\ = \;& \frac{1}{4}\left[\langle x+y, A(x+y)\rangle - \langle x-y, A(x-y)\rangle + i\langle x+iy, A(x+iy)\rangle - i\langle x-iy, A(x-iy)\rangle\right]\end{aligned}$$

As $A^* = A$, the first two terms are real, and the last two are imaginary. By replacing $y$ by $e^{i\varphi}y$, with a suitable $\varphi \in \mathbb{R}$, we can make the imaginary terms vanish, i.e., make $\langle e^{i\varphi}y, Ax\rangle$ real. This leads to the bound

$$\begin{aligned} 16|\langle y, Ax\rangle|^2 &\leq |\langle x+y, A(x+y)\rangle - \langle x-y, A(x-y)\rangle|^2 \\ &\leq a^2(\|x+y\|^2 + \|x-y\|^2)^2 \\ &= a^2(2\|x\|^2 + 2\|y\|^2)^2 = 16a^2 \end{aligned}$$

where we have used the definition of $a$, the Parallelogram Law, and the fact that $\|x\| = \|y\| = 1$. ∎

**Lemma 17.6** *Let $A = A^* \in \mathcal{B}(\mathcal{H})$, and suppose that $\lambda$ and $\mu$ are two disctinct eigenvalues of $A$ with eigenvectors $u$ and $v$ respectively. Then, $u \perp v$.*

**Proof:** By Lemma 16.7 we know that $\lambda$ and $\mu$ are real. Therefore,

$$\lambda\langle u, v\rangle = \langle Au, v\rangle = \langle u, Av\rangle = \mu\langle u, v\rangle$$

If $\lambda \neq \mu$, this implies $\langle u, v\rangle = 0$. ∎

Let $A$ be a linear operator on $\mathcal{H}$ and suppose $\mathcal{H}_0$ is a linear subspace of $\mathcal{H}$, satisfying $x \in \mathcal{H}_0$, for all $x \in \mathcal{H}_0$. $\mathcal{H}_0$ is then called an invariant subspace for $A$. If $A$ is bounded the closure of any invariant subspace is still an invariant subspace (why?).

**Lemma 17.7** *Let $A = A^* \in \mathcal{H}$, and let $\mathcal{H}_0$ be a closed invariant subspace for $A$. Then $\mathcal{H}_0^\perp$ is also an invariant subspace for $A$ (and, as it is an orthogonal complement, it is automatically closed).*

**Proof:** Let $x \in \mathcal{H}_0^\perp$, and $y \in \mathcal{H}_0$. Then, as $A = A^*$,

$$\langle y, Ax\rangle = \langle Ay, x\rangle = 0$$

where we have used that $Ay \in \mathcal{H}_0$. Therefore, $Ax \in \mathcal{H}_0^\perp$. ∎

In the situation of the lemma $\mathcal{H}$ has an orthogonal decomposition:

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$$

and the lemma implies that $A$ also decomposes into $A = A_0 + A_1$, where $A_0(x_0 \oplus x_1) = Ax_0$, and $A_1(x_0 \oplus x_1) = Ax_1$, for all $x_0 \in \mathcal{H}_0$, and $x_1 \in \mathcal{H}_0^\perp$.

We are now ready to proof the spectral theorem for compact self-adjoint linear operators.

**Proof: of Theorems 17.3 and 17.4**

*Step 1.*

First we will prove that if $A = A^*$ is compact, then either $\lambda = \|A\|$, or $\lambda = -\|A\|$, or both, is an eigenvalue of $A$. As there is nothing to prove if $A = 0$, suppose $A \neq 0$.

We start from the following expression for $\|A\|$ (Lemma 17.5):

$$\|A\| = \sup_{x, \|x\|=1} |\langle x, Ax\rangle|$$

which implies that we can find a sequence $(x_n)$, with $\|x_n\| = 1$, for all $n$, and such that

$$\lim_n |\langle x_n, Ax_n\rangle| = \|A\|$$

Since $A = A^*$, $\langle x_n, Ax_n\rangle$ is real for all $n$, there must be a subsequence of $(x_n)$, say $(y_n)$, such that

$$\lim_n \langle y_n, Ay_n\rangle = \lambda \tag{17.3}$$

with $\lambda = \|A\|$, or $\lambda = -\|A\|$. $(y_n)$ is a sequence of unit vectors and, therefore, bounded. The compactness of $A$ then guarantees the existence of a subsequence, which we denote again by $(y_n)$, such that $\lim_n Ay_n = z$ . We claim that $z$ is an eigenvector of $A$ with eigenvector $\lambda$. $z \neq 0$, since otherwise from (17.3) it would follow that $\lambda = 0$ and hence $\|A\| = |\lambda| = 0$, which is not the case since we assumed $A \neq 0$. That $z$ is an eigenvector can be checked as follows:

$$\begin{aligned}\|(A-\lambda)z\|^2 &= \lim_n \|(A-\lambda)Ay_n\|^2 = \|A(A-\lambda)y_n\|^2 \\ &\leq \lim_n \|A\|^2 \|(A-\lambda)y_n\|^2\end{aligned}$$

$$
\begin{aligned}
&= \lim_n \|A\|^2 \left[\|Ay_n\|^2 + \lambda^2\|y_n\|^2 - 2\lambda\langle y_n, Ay_n\rangle\right] \\
&\leq \lim_n \|A\|^2 \left[\|A\|^2\|y_n\|^2 + \lambda^2\|y_n\|^2 - 2\lambda\langle y_n, Ay_n\rangle\right] \\
&= \|A\|^2 \left[\lambda^2 + \lambda^2 - 2\lambda^2\right] = 0
\end{aligned}
$$

*Step 2.*

The idea is now to iterate Step 1 as follows. Denote the eigenvector found in Step 1 by $e_0$, and the corresponding eigenvalue by $\lambda_0$. Let $\mathcal{H}_0$ be the one-dimensional subspace of $\mathcal{H}$ generated by $e_0$, and consider the orthogonal decomposition of $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$$

It is obvious that $\mathcal{H}_0$ is an invariant subspace of $A$. As explained after Lemma 17.7, $A$ decomposes into $A = A_0 + A_1$, where $A_1$ is again compact (as the difference of two compact operators, Proposition 17.2). We also have that $\|A_1\| \leq \|A\|$:

$$\|A_1\| = \sup_{x\in\mathcal{H}_0^\perp, \|x\|=1} \|Ax\| \leq \sup_{x\in\mathcal{H}, \|x\|=1} \|Ax\| = \|A\|$$

Now, we can apply Step 1 to $A_1$ and find an eigenvector $e_1$ with eigenvalue $\lambda_1$, $|\lambda_1| = \|A_1\| \leq \|A\| = |\lambda_0|$. As $e_1 \in \mathcal{H}_0^\perp$ we have $A_0 e_1 = 0$, and

$$Ae_1 = A_0e_1 + A_1e_1 = \lambda_1 e_1$$

So, we have found a second eigenvalue of $A$, or, if $\lambda_1 = \lambda_0$, a second, independent, eigenvector belonging to $\lambda_0$.

We can now repeat this procedure : define $A_2$ to be the restriction of $A$ to the orthogonal complement of the two-dimensional subspace generated by $e_0$ and $e_1$. Then, $A_2$ is again compact and $\|A_2\| \leq \|A_1\|$. So, there will be an eigenvalue $\lambda_2$, $|\lambda_2| \leq |\lambda_1| \leq |\lambda_0|$. etc.

After $n$ iterations of this procedure, we will have found $n+1$ eigenvalues $\lambda_0, \ldots, \lambda_n$, in order of decreasing magnitude. If $A_{N+1} = 0$, for some $n$, we are done, and can write

$$A = \sum_{n=0}^{N} \lambda_n \langle e_n, \cdot\rangle e_n$$

The orthonormal set $\{e_n\}$ can be completed to an o.n. basis of $\mathcal{H}$, and all the other basis vectors will be eigenvectors of $A$ with eigenvalue 0.

If each $A_n \neq 0$, for all $n$, we obtain an infinite sequence of non-zero eigenvalues. We already know that the compactness of $A$ implies that non-zero eigenvalues have finite multiplicities, and also that

$$\lim_n \lambda_n = 0 \tag{17.4}$$

For any $N$ we have

$$A = \sum_{n=0}^{N} \lambda_n \langle e_n, \cdot \rangle e_n + A_{N+1}$$

where $A_{N+1}$ is 0 on the subspace generated by $e_0, \ldots, e_N$, and $\|A_{N+1}\| = |\lambda_{N+1}|$. By (17.4) we then have

$$\lim_N \|A - \sum_{n=0}^{N} \lambda_n \langle e_n, \cdot \rangle e_n\| = \lim_N |\lambda_{N+1}| = 0$$

In other words

$$A = \sum_{n=0}^{\infty} \lambda_n \langle e_n, \cdot \rangle e_n$$

where the sum converges in the operator norm. The vectors $e_n$ are o.n., but need not be complete. Denote their closed linear span by $\mathcal{H}'$, i.e.,

$$\mathcal{H}' = \{\sum_n c_n e_n \mid c_n \in \mathbb{C}, \sum_n |c_n|^2 < \infty\}$$

It is easy to see that any vector $x \in (\mathcal{H}')^{\perp}$, $Ax = A_N x$ for all $N$. Therefore,

$$\|Ax\| \leq \|A_N\| \, \|x\|, \quad \text{for all } N$$

As $\|A_N\| = |\lambda_n| \to 0$, this implies $Ax = 0$. So, $(\mathcal{H}')^{\perp} = \ker A$, and if so desired, we can complete $\{e_n\}$ to an o.n. basis by adding a basis of $\ker A$, consisting of zero-eigenvectors. ∎