Extra Exercises

Problem 1. Suppose $A \in \mathcal{B}(\mathcal{H})$ has an o.n. basis of eigenvectors $\{e_n\}_{n=0}^{\infty}$ with corresponding eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$.

a) Prove that A is normal.

b) Prove that A as above is unitary iff the eigenvalues λ_n satisfy $|\lambda_n| = 1$, for all $n \geq 0$.

c) Suppose $\lambda \in \mathbb{C} \setminus \overline{\{\lambda_n\}_{n=0}^{\infty}}$. Prove that $\lambda \mathbb{1} - A$ is invertible with a bounded inverse satisfying

$$\|(\lambda \mathbb{1} - A)^{-1}\| = \frac{1}{\inf_n |\lambda - \lambda_n|}$$

d) Note that b) also shows that $\sigma(A) = \overline{\{\lambda_n\}_{n=0}^{\infty}}$. Why?

e) Give example of an operator A that has an o.n. basis of eigenvectors and non-empty continuous spectrum.

Problem 2. Let f be the function defined by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \text{ for } z \in \mathbb{C}, |z| < R$$

where $R > 0$ is the radius of convergence of the power series, i.e.,

$$\frac{1}{R} = \limsup_n |a_n|^{1/n}$$

Consider $A \in \mathcal{B}(\mathcal{H})$, with $\|A\| < R$, and define $f(A)$ by through the power series. Prove that $f(A)^*$ is given by the power series

$$f(A)^* = \sum_{n=0}^{\infty} \overline{a_n} (A^*)^n$$

Problem 3. If A and $B \in \mathcal{B}(\mathcal{H})$ are self-adjoint, $\langle x, Ax \rangle$ and $\langle x, Bx \rangle$ are real numbers for all $x \in \mathcal{H}$. Therefore, it makes sense to define an order relation on the self-adjoint linear operators by

$$A \geq B \text{ iff } \langle x, Ax \rangle \geq \langle x, Bx \rangle, \quad \text{for all } x \in \mathcal{H}$$

Note that $A \geq 0$ is the same as A is positive definite, and $A \geq B$ iff $A - B \geq 0$. We often just say *positive* instead of *positive definite*. Of course, $A \leq B$ is equivalent to $B \geq A$.

a) Show that for any $A \in \mathcal{B}(\mathcal{H})$ the operators AA^* and A^*A are both positive.

b) Prove that if $A \geq B$ and $B \geq A$, then $A = B$.

c) It can be shown that if $A \geq 0$, $B \geq 0$, and $AB = BA$, then $AB \geq 0$. Give an example of two positive linear operators, A and B, such that AB is *not* positive.

d) Show that for any two self-adjoint operators, A and B, if $A \geq B$, then $C^*AC \geq C^*BC$, for all $C \in \mathcal{B}(\mathcal{H})$.

Problem 4. Let $A_n \in \mathcal{B}(\mathcal{H})$, $n = 1, 2, 3, \ldots$, be a sequence of self-adjoint operators satisfying

$$A_n \leq A_{n+1}, \qquad \text{for all } n \geq 1$$

and suppose there is $B = B^* \in \mathcal{B}(\mathcal{H})$ such that

$$A_n \leq B, \qquad \text{for all } n \geq 1$$

Suppose that all the operators A_n and B commute with each other. Prove that there exists $A = A^* \in \mathcal{B}(\mathcal{H})$ such that $A_n \to A$ in the strong operator topology. (Hint: Use the property stated in Problem 3 c).)

Problem 5. Let $A = A^* \in \mathcal{B}(\mathcal{H})$ be compact and such that A^2 is a projection.

a) Prove that A can be written as the difference of two orthogonal projections, i.e., there exist orthogonal projections P and Q, such that $A = P - Q$.

b) Prove that the projections P and Q as above can be chosen such that $A^2 = P + Q$.

c) Prove that if P and Q are orthogonal projections such that $A^2 = P + Q$, then P and Q are of finite rank.

Problem 6. Let $f : T \to \mathbb{C}$ be a C^∞ function, where T is the unit circle parametrized by the angle θ, and suppose there is a constant $a > 0$ such that

$$\|f^{(k)}\|_{L^2} \leq a^k, \qquad \text{for all } k \geq 0$$

where $f^{(k)}$ is the kth derivative of f with respect to θ. Prove that f is a trigonometric polynomial.