Assignment 7

(due date: Monday, March 2)

Problem 1. Let $\mathcal{H}$ be a Hilbert space and let (A_n) be a norm convergent sequence in $\mathcal{B}(\mathcal{H})$, with limit A. Prove that

$$A_n^* \to A^*, \quad \text{in norm.}$$

Problem 2. Let $\mathcal{H}$ be a Hilbert space and suppose $A \in \mathcal{B}(\mathcal{H})$ is invertible, i.e., there exists $A^{-1} \in \mathcal{B}(\mathcal{H})$ such that $AA^{-1} = \mathbb{1} = A^{-1}A$. Prove that A^* is also invertible and that

$$(A^{-1})^* = (A^*)^{-1}$$

Problem 3. For any $A \in \mathcal{B}(\mathcal{H})$, and $\lambda \in \rho(A)$, denote by R_λ the operator $(\lambda\mathbb{1} - A)^{-1}$. Prove that

a) $\rho(A^*) = \overline{\rho(A)} := \{\lambda \in \mathbb{C} \mid \overline{\lambda} \in \rho(A)\}$.

b) $R_\lambda(A)^* = R_{\overline{\lambda}}(A^*)$.

Problem 4. Let $\mathcal{H} = L^2([0,1])$ and let $k : [0,1] \times [0,1] \to \mathbb{C}$ be in $L^2([0,1] \times [0,1])$, i.e.,

$$\int_0^1 \int_0^1 |k(x,y)|^2 \, dx \, dy < +\infty$$

Define K by

$$Kf(x) = \int_0^1 k(x,y) f(y) \, dy$$

a) Show that K is a bounded linear operator on $L^2([0,1])$.

b) Show that K^* is also given as an integral operator of the form

$$K^* f(x) = \int_0^1 l(x,y) f(y) \, dy$$

and find its kernel l in terms of k.