Assignment 8

(due date: Monday, March 9)

Problem 1. Let $\mathcal{H}$ be a Hilbert space and suppose $A \in \mathcal{B}(\mathcal{H})$ is normal, i.e., $A^*A = AA^*$. Prove that the residual spectrum of A is empty.

Problem 2. Let $\mathcal{H} = l^2(\mathbb{Z})$. Define the right shift (operator) S on $\mathcal{H}$ by

$$S(x)_k = x_{k-1}, \qquad \text{for all } k \in \mathbb{Z} \tag{1}$$

where $x = (x_k)_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$.

a) Prove that S is invertible with bounded inverse and that $S^{-1} = S^*$.

b) prove that the spectrum of S is the unit circle, i.e., is given by

$$\sigma(S) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$$

c) Prove that $\sigma(S)$ consists entirely of continuous spectrum

Problem 3. Define an operator Δ on $l^2(\mathbb{Z})$ by

$$(\Delta x)_k = x_{k-1} - 2x_k + x_{k+1}, \text{ for all } x = (x_k)_{k\in\mathbb{Z}} \in l^2(\mathbb{Z}) \ .$$

Prove that the spectrum of Δ is entirely continuous, real, and given by the interval $[-4, 0]$. (Hint: Note that $\Delta = S + S^* - 2\mathbb{1}$.)

Problem 4. Define the left shift L on $l^2 := l^2(\{0, 1, 2, 3, \ldots\})$ by

$$L(x) = (x_1, x_2, \ldots) \text{ for all } x = (x_0, x_1, x_2, \ldots) \in l^2 \quad .$$

and the right shift R on $l^2 := l^2(\{0, 1, 2, 3, \ldots\})$ by

$$R(x) = (0, x_0, x_2, \ldots) \text{ for all } x = (x_0, x_1, x_2, \ldots) \in l^2 \quad .$$

a) Compute the adjoints L^* and R^* of the operators L and R, respectively.

b) Describe the operators R^*R and L^*L.