Assignment 9

(due date: Monday, March 16)

Problem 1. Let $\mathcal{H}$ be a separable Hilbert space, and let $\{e_n\}_{n=1}^{\infty}$ be an o.n. basis of $\mathcal{H}$. For all $N \geq 1$, define $P_N \in \mathcal{B}(\mathcal{H})$ by

$$P_N x = \sum_{n=1}^{N} \langle e_n, x \rangle e_n, \quad \text{for all } x \in \mathcal{H}$$

Prove that if $A \in \mathcal{B}(\mathcal{H})$ is compact, then the sequence $P_N A$ converges to A in the operator norm topology.

Problem 2. Define an operator $K \in \mathcal{B}(L^2([0,\pi])$ by

$$Kf(x) = \frac{2i}{\pi} \int_0^{\pi} \sin 2(x-y) f(y)\, dy, \quad \text{for all } f \in L^2([0,\pi])$$

Let $\mathcal{H}_0$ be the two-dimensional subspace of $L^2([0,\pi])$ spanned by the functions $\cos 2x$ and $\sin 2x$.

a) Show that K is self-adjoint.

b) Prove that K is compact.

c) Prove that $\mathcal{H}_0^{\perp} \subset \ker K$.

d) Diagonalize K, i.e., find its eigenvalues and an o.n. basis of eigenvectors of K.

e) Determine $\|K\|$.