20 Unbounded linear operators and Green's functions

We have noted before that differential operators on a (non-trivial) normed function space cannot be bounded. Clearly, differential operators are important for the study of differential equations and we would like to study them in detail in spite of their lack of continuity. There are two main approaches to this problem, which are the main themes of this quarter. One is to consider differential operators as acting on topological function spaces (necessarily not normed) designed such that the relevant operators become continuous. The other approach, which we will follow in this chapter, is to consider special classes of unbounded operators (e.g., self-adjoints) defined on a dense linear subspace of a Hilbert space. Often, the important differential operators will have bounded inverses (defined in terms of their "Green's function") and this will help in the analysis.

20.1 Definitions: domain, adjoint, selfadjoint

We will restrict ourselves to linear operators defined on a dense linear subspace of a Hilbert space $\mathcal{H}$, although unbounded operators on a Banach space can be considered along the same lines.

The definition of a (not necessarily bounded) linear operator A on $\mathcal{H}$ includes the definition of its *domain* $\mathcal{D}(A)$, which is assumed to be a linear subspace of $\mathcal{H}$. So, a linear operator is a linear map $A : \mathcal{D}(A) \to \mathcal{H}$. If $\mathcal{D}(A)$ is *dense* in $\mathcal{H}$, we say that A is *densely defined.* We will usually assume that $\mathcal{D}(A)$ is dense. If it is not, we can always define A to be 0 on $\mathcal{D}(A)^\perp$. So, this is without loss of generality.

Example: Let L be the differential operator with domain $\mathcal{D}(L) = C^2([0,1]) \subset L^2([0,1])$, and $Lf = f''$, for all $f \in \mathcal{D}(L)$. As we will see lateron, there are other, more useful, choices of the domain on which to define the second derivative as an operator on L^2.

An operator B is called an *extension* of A if $\mathcal{D}(A) \subset \mathcal{D}(B)$, and $Bx = Ax$, for all $x \in \mathcal{D}(A)$.

Theorem 20.1 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, and let A be a linear map $\mathcal{D}(A) \to \mathcal{H}_2$ with $\mathcal{D}(A)$ a linear subspace of $\mathcal{H}_1$. If A is bounded with norm (least bound) $\|A\|$, then there is a unique bounded extension $\tilde{A}$ of A with $\mathcal{D}(\tilde{A}) = \overline{\mathcal{D}(A)}$, and one has $\|\tilde{A}\| = \|A\|$.*

Note: For this reason if $\mathcal{D}(A)$ is dense and A is bounded we can equally well define A on all of $\mathcal{H}_1$, i.e., A is a bounded linear transformation $\mathcal{H}_1 \to \mathcal{H}_2$.

The *adjoint* of a linear operator A with dense domain is defined as follows. For any $y \in \mathcal{H}$ consider the linear functional on $\mathcal{D}(A)$ defined by

$$\varphi_y(x) = \langle y \mid Ax \rangle$$

As $\mathcal{D}(A)$ is assumed to be dense, by the previous theorem, if φ_y is bounded it can be uniquely extend to a bounded linear functional on $\mathcal{H}$. By the Riesz Representation Theorem we then have a unique vector $z \in \mathcal{H}$ satisfying

$$\langle y, Ax \rangle = \langle z, x \rangle, \qquad \text{for all } x \in \mathcal{D}(A)$$

The adjoint of A, denoted by A^*, is then the linear operator with

$$\mathcal{D}(A^*) = \{y \in \mathcal{H} \mid \varphi_y \text{ is bounded }\}$$

and $A^* y = z$ for all $y \in \mathcal{D}(A^*)$.

As before, we call an operator A *self-adjoint* if $A^* = A$, but now this includes the equality of their domains: $A^* = A$ iff $\mathcal{D}(A^*) = \mathcal{D}(A)$, and $A^* x = Ax$, for all $x \in \mathcal{D}(A)$.

Note that it is possible that A^* exists but one cannot define A^{**} because $\mathcal{D}(A^*)$ is not dense.

Exercise 20.1 *Prove that if A^{**} exists, then it is an extension of A.*

Examples:
1) Take $\mathcal{H} = l^2(\mathbb{N})$, and define an operator A as follows:

$$\mathcal{D}(A) = \{(z_n)_{n=1}^{\infty} \in l^2(\mathbb{N}) \mid \text{ only finitely many of the } z_n \neq 0\}$$

and

$$A(z_1, z_2, z_3, \ldots) = (z_1, 2z_2, 3z_3, \ldots)$$

Then A^* is an extension of A with domain

$$\mathcal{D}(A^*) = \{(z_n)_{n=1}^{\infty} \in l^2(\mathbb{N}) \mid \sum_{n=1}^{\infty} n^2 |z_n|^2 < \infty\}$$

A^* is self-adjoint, i.e., $A^{**} = A^*$.
2) Take $\mathcal{H} = L^2(T^1)$. Define an operator A with domain

$$\mathcal{D}(A) = H^2(T^1) = \{f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx} \mid \sum_{n\in\mathbb{Z}} n^4 |a_n|^2 < \infty\}$$

and, for any $f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$,

$$Af(x) = \sum_{n\in\mathbb{Z}} n^2 a_n e^{inx}$$

A is just one of the possible definitions of $-d^2/dx^2$ as an operator on $L^2(T^1)$. We now verify that A is self-adjoint. For any $g \in \mathcal{D}(A^*)$, $g(x) = \sum_n b_n e^{inx}$, we should have

$$\langle g, Af \rangle = \sum_n \overline{b_n} n^2 a_n$$

which is a bounded linear functional of f iff $(n^2 b_n) \in l^2(\mathbb{Z})$, due to the Riesz Representation Theorem, i.e., iff $g \in \mathcal{D}(A)$. This proves $\mathcal{D}(A^*) = \mathcal{D}(A)$. Clealry we also have $A^* g = Ag$ for all $g \in \mathcal{D}(A)$.

Note that A is also an extension of $-d^2/dx^2$ with domain $C^2(T^1)$.

We will come back to the problem of finding self-adjoint extensions (restrictions) of more general differential operators.

20.2 Inverses

One is often interested in solving linear equations of the form

$$Ax = y \tag{20.1}$$

for given y.

If $\dim \mathcal{H} < \infty$ the conditions for unique solvability of (20.1) are given by the so-called *Fredholm Alternative*:

A. $\ker A^* = \{0\}$. This is equivalent to $\ker A = \{0\}$ (finite dimension! See Exercise 20.3), in which case A is invertible, i.e., there is a unique linear map A^{-1} such that $AA^{-1} = \mathbb{1} = A^{-1}A$, and, hence, (20.1) is solved by $x = A^{-1}y$, for all $y \in \mathcal{H}$.

B. $\ker A^* \neq \{0\}$. (20.1) can be solved iff $y \in \mathrm{ran}A$, by the definition of $\mathrm{ran}A$. As $\mathrm{ran}A = (\ker A^*)^\perp$, this can be expressed as $\langle z, y\rangle = 0$ for all z such that $A^*z = 0$. In this case, however, the solution is not unique: if x_0 is a solution, i.e, $Ax_0 = y$, then all $x \in x_0 + \ker A$ are solutions, too (and these are exactly all solutions).

Exercise 20.2 *Prove that, if* $\dim \mathcal{H} < \infty$, $\dim\ker A + \dim \mathrm{ran}A = \dim \mathcal{H}$.

Exercise 20.3 *Prove that, if* $\dim \mathcal{H} < \infty$, $\dim\ker A = \dim\ker A^*$. *In particular,* $\ker A = \{0\}$ *iff* $\ker A^* = \{0\}$.

If $\dim \mathcal{H} = \infty$, things are not so simple. We study this problem in detail for differential operators in the next section.

As a matter of illustration, do the following exercise:

Exercise 20.4 *Consider the right shift S on $\mathcal{H} = l^2(\mathbb{N})$. Prove that* $\ker S = \{0\}$, *but* $\mathrm{ran}S$ *is not dense in $\mathcal{H}$. Hence, there cannot be a densely defined inverse S^{-1} such that $SS^{-1} = \mathbb{1}$.*

20.3 Differential operators

A linear differential operator of order n is a linear map L defined on a suitable function space that acts on $f \in C^n$ by

$$Lf = \sum_{j=0}^{n} a_j(x) f^{(j)}(x)$$

Here, $f^{(j)}$ denotes the jth derivative of f, and the $a_j(x)$ are complex-valued functions. For concreteness we assume that all functions are defined of the interval $[0,1]$. If the coefficients a_j are polynomials, then L maps polynomials into polynomials. If the coefficients are C^∞ functions, L transforms C^∞ into itself. But it is often necessary to consider L as an operator defined on various other dense domains in L^2.

Our goal is to study ODE's of the form

$$Lu = f \tag{20.2}$$

where we want to find the solutions u for given f. First, we will look for conditions for unique solvability of (20.2) (conditions for the existence of a unique Green's function). Then we will learn how to find, i.e., construct, the Green's function and the solutions.

20.3.1 The Green's function for boundary value problems

In this section, we will limit ourselves to 2nd order differential operators L acting on function $u : [0,1] \to \mathbb{C}$:

$$Lu = au'' + bu' + cu \tag{20.3}$$

Assume that $a(x) > 0$ for all $x \in [0,1]$, and that a, b, and c are continuous. Then, the ODE

$$Lu = 0$$

has a two-dimensional space of C^2 solutions

$$u = \alpha_1 u_1 + \alpha_2 u_2, \quad \alpha_1, \alpha_2 \in \mathbb{C}$$

Obviously, if we do not impose extra conditions on the solutions, (20.3) will have at least a two-dimensional space of solutions if it has a solution at all. Unique solutions are obtained by imposing boundary conditions, e.g.,

Dirichlet:

$$u(0) = u(1) = 0,$$

Neumann:

$$u'(0) = u'(1) = 0,$$

periodic:

$$\begin{aligned} u(0) &= u(1) \\ u'(0) &= u'(1), \end{aligned}$$

mixed:

$$\begin{gathered} \alpha_0 u(0) + \beta_0 u'(0) = 0 \\ \alpha_1 u(1) + \beta_1 u'(1) = 0. \end{gathered}$$

where $\alpha_0, \alpha_1, \beta_0, \beta_1$ are complex constants.

Instead of imposing conditions at both endpoints, one can also impose two conditions at one of the endpoints. E.g., $u(0) = u'(0) = 0$. This is then usually called *initial conditions*.

The boundary conditions given above are *homogeneous* boundary conditions. For linear ODE, non-homogeneous boundary conditions can be reduced to homogeneous ones by defining $v(x) = u(x) - u_0 - (u_1 - u_0)x$. If v is then a solution of the homogeneous problem

$$\begin{aligned} Lv &= f \\ v(0) &= 0 \\ v(1) &= 0 \end{aligned}$$

u will solve the non-homogeneous system

$$\begin{aligned} Lu &= f \\ u(0) &= u_0 \\ u(1) &= u_1 \end{aligned}$$

Before we commence the rigorous discussion of Green's functions of differential operators + boundary conditions, we first explain the basic idea behind it. For concreteness, let us consider Dirichlet boundary condtions, i.e., we want to solve

$$\begin{aligned} Lu &= f \\ u(0) &= 0 \\ u(1) &= 0 \end{aligned} \tag{20.4}$$

As the inverse operator of taking the derivative can be expressed as an integral operator with kernel function $\mathrm{Ind}[y \leq x]$, it is natural to try to find the inverse of a differential operator in the form of an integral operator. I.e., we look for a function $g : [0,1] \times [0,1] \to \mathbb{C}$ such that the operator G defined by

$$Gf(x) = \int_0^1 g(x,y)f(y)\,dy \tag{20.5}$$

is the inverse of L, i.e.,

$$LG = \mathbb{1}, \qquad \text{on } L^2([0,1]) \tag{20.6}$$

(and $GL = \mathbb{1}$ on $\mathcal{D}(L)$).

Using (20.5) and (20.6) we obtain the following condition on g:

$$L\int_0^1 g(x,y)f(y)\,dy = f(x), \qquad \text{for all } f \tag{20.7}$$

In order to have Gf satsify the Dirichlet boundary conditions we also require that, for all $y \in [0,1]$, $g(0,y) = g(1,y) = 0$. Then, for any f, the solution u of (20.4) can be written as

$$u(x) = \int_0^1 g(x,y)f(y)\,dy$$

There are several ways of rewriting the condition (20.7). One is to make use of the notion of delta function, and write a differential equation for g in distributional sense:

$$Lg(x,y) = \delta(x-y)$$

Here we consider L to be acting on g as a function of its first variable.

We will study distributions later, but in any case, at first, it is preferable to formulate an ordinary ODE for g.

Proposition 20.2 *Let L be given by (20.3) with $a \in C^1([0,1])$, $a(x) > 0$, for all $x \in [0,1]$, and $b, c \in C([0,1])$. Suppose $g : [0,1] \times [0,1] \to \mathbb{C}$ is continuous and, for all $y \in [0,1]$, g is C^1 in x on $x < y$ and $x > y$, and such that*

$$Lg = 0, \qquad \text{for all } x \neq y \tag{20.8}$$

$$a(y)\lim_{\epsilon \downarrow 0}\left[g_x(x, x+\epsilon) - g_x(x, x-\epsilon)\right] = 1 \tag{20.9}$$

$$g(0,y) = g(1,y) = 0 \tag{20.10}$$

Then G given by (20.5) satisfies $LG = \mathbb{1}$ on $L^2([0,1])$, and for any $f \in C([0,1])$, Gf satisfies the Dirichlet boundary conditions and is a solution of (20.4).

Proof: The proof is by direct computation. The only non-trivial part is to check that

$$u(x) := \int_0^1 g(x,y)f(y)\,dy$$

satisfies $Lu = f$. From the ODE, and the fact that a is continuous and striclty positive, it is easy to see that g will be C^2 where it is C^1. $g_x x$ is therefore a.e. well-defined, and, ignoring the arguments where it is not, $g_x x$ is integrable as a function of two variables.

$$\begin{aligned} Lu &= L\int_0^1 g(x,y)f(y)\,dy \\ &= \left[a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)\right]\left[\int_0^x g(x,y)f(y)\,dy + \int_x^1 g(x,y)f(y)\,dy\right] \end{aligned}$$

To compute the derivatives we use Lemma 20.3 given below. The result is:

$$\begin{aligned} &\int_0^1 Lg(x,y)f(y)\,dy + a(x)[g_x(x,x-0)f(x-0) - g_x(x,x+0)]f(x+0) \\ &= 0 + a(x)[g_x(x+0,x) - g_x(x-0)]f(x) = f(x) \end{aligned}$$

by the assumptions. ■

Lemma 20.3 *If $h(x,y)$ is a continuous function of two variables, and has an integrable almost everywhere defined partial derivative with repsect to x, and a and b are C^1, then*

$$\frac{d}{dx}\int_{a(x)}^{b(x)} h(x,y)\,dy = b'(x)h(x,b(x)) - a'(x)h(x,a(x)) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}h(x,y)\,dy$$

Exercise 20.5 *Prove Lemma 20.3.*

20.3.2 Construction of the Green's function

We start from some basic facts about the solutions of linear 2nd order ODE. When a, b, c, f are continuous functions and $a(x) \neq 0$ on an interval (a,b) containing x_0, the homogeneous ODE

$$au'' + bu' + cu = 0$$

has a two-dimensional space of solutions, i.e., spanned by two linearly independent solutions. As long as $a(x)$ does not vanish the Wronskian of two independent solitions does not vanish either:

$$W(x) = \det\begin{pmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_1'(x) \end{pmatrix} \neq 0$$

Also, under the same assumptions, the ODE with b.c.

$$au'' + bu' + cu = f \tag{20.11}$$

$$u(x_0) = U, u'(x_0) = V \tag{20.12}$$

has a unique solution $u(x)$ on the interval (a, b).

In order to construct g satisfying the conditions of Proposition 20.2 we start from a pair of linearly independent solutions v_0 and v_1 such that

$$v_0(0) = 0, \text{ and } v_1(1) = 0$$

Such a pair exists iff the homogeneous Dirichlet problem, i.e., (20.11-20.12) with $f = U = V = 0$, has only the solution $u \equiv 0$.

Exercise 20.6 *Prove the preceding statement.*

The Green's function can now be found in the following form:

$$g(x,y) = C(y) \times \begin{cases} v_0(x)v_1(y) & \text{if } x < y \\ v_0(y)v_1(x) & \text{if } x > y \end{cases} \tag{20.13}$$

The function $C(y)$ is determined by (20.9):

$$C(y) = \frac{1}{a(y)W(y)}$$

It is then clear that g is continuous, satisfies $Lg = 0$ whenever $x \neq y$, and $g(0, y) = g(1, y) = 0$.

To summarize, we found that, if the homogeneous Dirichlet problem has only the zero solution, g defined by has all the properties required in Proposition 20.2.

Example: The stationary temperature distribution in a rod of unit length, with both ends kept at a constant temperature of 0 degrees, with heat loss through its surface given by $\alpha^2 u$, and subject to a non-uniform heat source desrbed by f, is the solution of

$$-u'' + \alpha^2 u = f \tag{20.14}$$

For the construction of the Green's function we need two linearly independent solutions of (20.14) with $f = 0$, v_0 and v_1, satisfying $v_0(0) = 0$, and $v_1(1) = 0$. The general solution of the homogeneous equation is of the form

$$c_+ e^{\alpha x} + c_- e^{-\alpha x}$$

For v_0 and v_1 we choose

$$\begin{aligned} v_0(x) &= \frac{e^{\alpha x} - e^{-\alpha x}}{2} = \sinh \alpha x \\ v_1(x) &= \frac{-e^{-\alpha} e^{\alpha x} + e^{\alpha} e^{-\alpha x}}{2} = \sinh \alpha(1-x) \end{aligned}$$

the Wronskian turns out to be a constant $W \equiv \alpha \sinh \alpha$. The Green's function is then

$$g(x,y) = \begin{cases} \frac{\sinh \alpha x \sinh \alpha(1-y)}{\alpha \sinh \alpha} & \text{if } 0 \leq x \leq y \leq 1 \\ \frac{\sinh \alpha y \sinh \alpha(1-x)}{\alpha \sinh \alpha} & \text{if } 0 \leq y \leq x \leq 1 \end{cases}$$

20.3.3 Self-adoint differential operators and Sturm-Liouville operators

In this section we will consider self-adjoint differential operators and their Green's functions in some more detail. We start with a formula due to Green for the action of the adjoint L^* of a differential operator L on sufficiently smooth functions (C^n functions if n is the order of the differential operator).

Lemma 20.4 *Let L be a linear operator such that for all $f \in \mathcal{D}(L) \cap C^n([0,1])$, $Lf = (d/dx)^n f$. Then, for all $f, g \in \mathcal{D}(L) \cap C^n([0,1])$,*

$$\langle g, Lf \rangle = (-1)^n \langle Lg, f \rangle + \left[g f^{(n-1)} - g^{(1)} f^{(n-2)} + \cdots + (-1)^{n-1} g^{(n-1)} f \right]_0^1$$

where, for any function h, $h^{(k)}$ denotes the kth derivative of h.

Proof: The proof is nothing more than integration by parts. ■

In particular the operator $D = d/dx$ on the domain

$$\mathcal{D}(D) = \{ f \in C^1([0,1]) \mid f(0) = f(1) = 0 \}$$

satisfies $\mathcal{D}(D) \subset \mathcal{D}(D^*)$ and for all $f \in \mathcal{D}(D)$

$$D^* f = -Df$$

And similarly for $D^2 = (d/dx)^2$ on the domain

$$\mathcal{D}(D^2) = \{ f \in C^2([0,1]) \mid f(0) = f(1) = 0 \}$$

satisfies $\mathcal{D}(D^2) \subset \mathcal{D}((D^2)^*)$ and for all $f \in \mathcal{D}(D^2)$

$$(D^2)^* f = D^2 f$$

By combining these relations with the following exercises we can compute the action of the adjoint of any 2nd order differential operator with sufficiently smooth coefficients on sufficiently smooth functions.

Exercise 20.7 *Let A and B be two linear operator on a Hilbert space $\mathcal{H}$ with domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$ respectively, and assume $\mathcal{D}(A) \cap \mathcal{D}(B) \neq \emptyset$. Define an operator C by $\mathcal{D}(C) = \mathcal{D}(A) \cap \mathcal{D}(B)$ and $Cx = Ax + Bx$, for all $x \in \mathcal{D}(C)$. Prove that C^* is an extension of $A^* + B^*$.*

Exercise 20.8 *Let A and B be two linear operator on a Hilbert space $\mathcal{H}$ with domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$ respectively. Define $\mathcal{D}(AB)$ and $\mathcal{D}(B^*A^*)$ by*

$$\begin{aligned} \mathcal{D}(AB) &= \{ x \in \mathcal{D}(B) \mid Bx \in \mathcal{D}(A) \} \\ \mathcal{D}(B^*A^*) &= \{ x \in \mathcal{D}(A^*) \mid A^*x \in \mathcal{D}(B^*) \} \end{aligned}$$

and assume that $\mathcal{D}(AB)$ and $\mathcal{D}(B^*A^*)$ are non-empty. Define operators AB and B^*A^* on their respective domains in the obvious way. Prove that $(AB)^*$ is an extension of B^*A^*.

Proposition 20.5 (Green's formula) *Define $\mathcal{D}(L)$ by*

$$\mathcal{D}(L) = \{u \in C^2([0,1]) \mid u(0) = u(1) = 0\}$$

Let $a \in C^2([0,1]), b \in C^1([0,1]), c \in C([0,1])$ and define

$$Lu = au'' + bu' + cu, \qquad \textit{for all } u \in \mathcal{D}(L).$$

Then $\mathcal{D}(L) \subset \mathcal{D}(L^*)$ and

$$L^*u = (u\overline{a})'' - (u\overline{b})' + u\overline{c}, \qquad \textit{for all } u \in \mathcal{D}(L).$$

Proof: By Exercise 20.7 we can compute the action of L^* term by term. E.g., for the first term, apply Exercise 20.8 with $A = (d/dx)^2$ and B is multiplication by the complex-valued function a. B^* is then multiplication by the complex-valued function $\overline{a}$, and the result follows. ■

Next, we address the question which 2nd order differential operators are self-adjoint. Recall that $L^* = L$ means $\mathcal{D}(L^*) = \mathcal{D}(L)$ and $L^*u = Lu$, for all $u \in \mathcal{D}(L)$. First, we investigate the conseqences of the latter requirement:

$$\begin{aligned} L^*u &= (u\overline{a})'' - (u\overline{b})' + u\overline{c} \\ &= \overline{a}u'' + (2\overline{a}' - \overline{b})u' + (\overline{a}'' - \overline{b}' + \overline{c})u \\ &= au'' + bu' + cu \end{aligned}$$

As u, u', and u'', are in general linearly independent, we must therefore have

$$\begin{aligned} a &= \overline{a} \\ b &= 2\overline{a}' - \overline{b} \\ c &= \overline{a}'' - \overline{b}' + \overline{c} \end{aligned}$$

These relations are easily reduced to the following conditions:

$$\begin{aligned} &a \text{ is real} \\ &\mathrm{Re}\ b = a' \\ &\mathrm{Im}\ c = -\frac{1}{2}\mathrm{Im}\ b \end{aligned}$$

where, Re z and Im z denote the real and imaginary parts of a complex number z, respectively. So, the coefficients of a self-adjoint L are determined by three real functions: a, Im b, and Re c. If one restricts to real coefficients, there are only two real functions as independent parameters, traditional called p and q:

$$\begin{aligned} a &= -p \\ b &= -p' \\ c &= q \end{aligned}$$

In this way we arrive at the standard form for a *Sturm-Liouville* operator:

$$Lu = -(pu')' + qu$$

20.4 The spectral Theorem for Sturm-Liouville operators

The standard Sturm-Liouville problem we considered so far is determined by two real functions, p and q, about which we assume that $p \in C^1([0,1])$, $q \in C([0,1])$, $p(x) > 0$ for all $x \in [0,1]$. The corresponding eigenvalue problem is then:

$$\begin{aligned} Lu := -(pu')' + qu &= \lambda u \\ u \in \mathcal{D}(L) &= \{u \in H^2([0,1]) \mid u(0) = u(1) = 0\} \end{aligned} \tag{20.15}$$

If λ_0 is not an eigenvalue, i.e., the above problem with $\lambda = \lambda_0$ has only the $u \equiv 0$ solution, then by the previous paragraphs, the Green's function g of $L - \lambda_0 \mathbb{1}$ exists and is unique, and we can define

$$Gu(x) = \int_0^1 g(x,y)u(y)\,dy; \qquad G : L^2 \to \mathcal{D}(L)$$

As g is continuous, real, and symmetric, G is self-adjoint and compact, and hence has a o.n. basis $\{u_n\}$ of eigenvectors, with eigenvalues g_n. As $LG = \mathbb{1}$ we must have that $u_n \in \mathcal{D}(L)$, and u_n is and eigenvector of L with eigenvalue λ_n satisfying $\lambda_n - \lambda_0 = g_n^{-1}$. It is also easy to see that (20.15) cannot have other linearly independent solutions. Thus we have proved a spectral theorem for Sturm-Liouville operators.

20.5 Singular Sturm-Liouville problems

The Sturm-Liouville operators considered above are called *regular*. In the construction of the Green's function we have used that p (or a) does not vanish. If p does vanish somewhere in the bounded interval one is considering, or if the interval is unbounded, the Sturm-Liouville problem is said to be *singular*. For singular problems it can be more difficult to prove that G is compact, or even it may not be true. Here, we limit ourselves to a few examples.

Example 1: $p(x) = (x^2 - 1)$ vanishes at the boundaries of the interval $[-1,1]$. The corresponding S-L eigenvalue problem has the Legendre polynomials as its solutions.

Example 2: An example of a problem on the whole of $\mathbb{R}$ with a compact G is the quantum harmonic oscillator:

$$\begin{aligned} Lu &= -u'' + x^2 \\ \mathcal{D}(L) &= \{u \in H^2(\mathbb{R}) \mid \int_{\mathbb{R}} x^4 |u(x)|^2\,dx < \infty\} \end{aligned}$$

Its eigenvectors are the Hermite functions. As the corresponding G is self-adjoint and compact we obtain for free that the Hermite functions as an orthormal set in $L^2(\mathbb{R})$ are complete.

Example 3: Finally, here is a Sturm-Liouville operator on $L^2(\mathbb{R})$ with a non-compact inverse:

$$\begin{aligned} Lu &= -u'' + u \\ \mathcal{D}(L) &= H^2(\mathbb{R}) \end{aligned}$$

Its Green's operator is given by

$$Gu(x) = \int_{\mathbb{R}} e^{-|x-y|} u(y)\, dy$$

In fact it is not hard to see that the spectrum of L (and G) is $[1, +\infty)$.

20.6 Applications of Sturm-Liouville problems

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We briefly mention some applications of Green's functions and Sturm-Liouville problems. We refer to the literature for many more applications in the theory of PDE.

20.6.1 PDE: initial value problems

This is the application we mentioned at the beginning of this chapter as our main motivation for the study of the eigenvalue problem for unbounded operators. A PDE initial value problem of the form

$$\begin{aligned} u_t &= Lu \\ u(x,0) &= f(x) \end{aligned}$$

$x \in [a,b]$, $t \in \mathbb{R}$, has solutions of the form

$$u(x,t) = \sum_n e^{\lambda_n t} \langle u_n, f \rangle e_n(x)$$

for f and t for which this defines a differentiable function.

20.6.2 An application in higher dimensions: elliptic boundary value problems

Higher-dimensional problems can often be reduced to lower-dimensional ones by separation of variables and the use of symmetry. But Green's functions can also be constructed for higher-dimensional problems directly. In such case we are strictly speaking no longer delaing with Sturm-Liouville problems. The main issues, however, are the same: self-adjointness, compactness of the inverse operator etc..

We limit ourselves to a particular example: the Dirichlet problem on a bounded domain Ω with smooth boundary in $\mathbb{R}^n$:

$$Lu := -\Delta u = \lambda u \tag{20.16}$$
$$\mathcal{D}(L) = \{u \in H^2(\Omega) \mid u(x) = 0, x \in \partial\Omega\} \tag{20.17}$$

where $\partial\Omega$ denotes the boundary of Ω (i.e., the difference between the smallest closed set in $\mathbb{R}^n$ containing it, and the largest open set contained in it). We will return to this example later and prove that indeed the inverse of L is given by a compact self-adjoint Green's operator. As a consequence there is an o.n. basis of $L^2(\Omega)$ consisting of eigenvectors of the Laplacian and satisfying Dirichlet boundary conditions. Other boundary conditions and more general operators can be considered as well.