23 Applications of the Fourier Transform

23.1 Translation Invariance (Symmetry)

Consider an integral operator K on $L^2(\mathbb{R})$ (L^2, for short), defined by

$$Kf(x) = \int_{-\infty}^{+\infty} k(x,y)f(y)\,dy \tag{23.1}$$

Suppose there is a function $g \in L^2 \cap L^1$, satisfying $g(-x) = \overline{g(x)}$, and such that $k(x,y) = g(x-y)$. It follows that K is a self-adjoint bounded linear operator on L^2 and that it commutes with translations, which are represented by the unitary operators T_a:

$$KT_a = T_a K, \quad \text{for all } a \in \mathbb{R}$$

If K and T_a were two finite-dimensional matrices, we would conclude that there is a basis of common eigenvectors of T_a and K. Note that K of the form (23.1) cannot be Hilbert-Schmidt. The Fourier transform, considered as a unitary $\mathcal{F} : L^2 \to L^2$, allows us the directly compute the spectrum of K. Define $\hat{K} = \mathcal{F}^{-1} K \mathcal{F}$. Then

$$\hat{K}\hat{f}(k) = \sqrt{2\pi}\hat{g}(k)\hat{f}(k)$$

As $\hat{f}$ is an arbitrary element of L^2, this means that $\hat{K}$ is the multiplication operator by the function $\sqrt{2\pi}\hat{g}$. As we assumed that $g \in L^1$, $\hat{g} \in C_0$, by the Riemann-Lebesgue Lemma (Theorem 22.3). Just as with the operator of multiplication by x, on $L^2([0,1])$, that we studied last quarter, the spectrum of $\hat{K}$ is exactly the closure of the range of values taken by the function $\hat{g}$. As K and $\hat{K}$ are unitarily equivalent ($\hat{K} = \mathcal{F}^{-1} K \mathcal{F}$), this is then also the spectrum of K.

Exercise 23.1 *Prove that if $g \in L^2$ satisfies $g(-x) = \overline{g(x)}$, then $\hat{g}$ is real-valued.*

Exercise 23.2 *Let G be a multiplication operator on $L^2(\mathbb{R})$, defined by*

$$Gf(x) = g(x)f(x)$$

where g is continuous and bounded. Prove that G is a bounded linear operator on $L^2(\mathbb{R})$ and that its spectrum is given by

$$\sigma(G) = \overline{\{g(x) \mid x \in \mathbb{R}\}}$$

Exercise 23.3 *Can an operator G of the form given in the previous exercise have any eigenvalues? Give a (counter)example.*

For concreteness consider the following example:

$$Kf(x) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-y)^2} f(y)\, dy$$

Then

$$\hat{K}\hat{f}(k) = e^{-\frac{1}{2}k^2}\hat{f}(k)$$

By the discussion above we then have $\sigma(K) = \sigma(\hat{K}) = [0,1]$. It is not hard to prove that this is entirely continuous spectrum. So, there are no eigenvectors. There are, however, generalized eigenvectors, in the sense of distributions: for any $l \in \mathbb{R}$

$$\hat{K}\delta_l = e^{-\frac{1}{2}l^2}\delta_l$$

By taking Fourier transforms we find for K that

$$Ke^{-ilx} = e^{-\frac{1}{2}l^2}e^{-ilx}$$

where the functions are to be interpreted as the distributions they define via the pairing. As one expects from the fact that K and T_a commute, these are also "eigendistributions" of T_a:

$$T_a e^{-lx} = e^{-il(x-a)} = e^{ial}e^{-ilx}$$

The corresponding "eigenvalues" of the unitary T_a are e^{ial}. So, if $a \neq 0$, $\sigma(T_a) = S^1$, and it is entirely continuous spectrum of T_a, considered as an operator on L^2. For $a = 0$, we have $T_0 = \mathbb{1}$, whose spectrum contains just the eigenvalue 1.

A concrete application of the integral operator K is in the study of the heat equation on the real line (an "infinite rod"):

$$\begin{aligned} u_t &= \frac{1}{2}u_{xx}, \qquad \text{for all } x \in \mathbb{R}, t > 0 \\ \lim_{x\to\pm\infty} u(x,t) &= 0, \qquad \text{for all } t > 0 \\ \lim_{t\downarrow 0} u(x,t) &= u_0(x), \qquad \text{for all } x \in \mathbb{R} \end{aligned}$$

Here, u_0 is the initial condition, which we can allow to be a distribution satisfying certain conditions. The solution is then given by

$$u(x,t) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2t}(x-y)^2}}{\sqrt{2\pi t}} u_0(y)\, dy$$

One immediately sees that for any initial condition $u_0 \in L^1$, the solution is C^∞ in both t and x, for $t > 0$ and $x \in \mathbb{R}$. This is the well-known *smoothing property* of the heat equation.

One can also consider a half-infinite rod, i.e., the heat equation on $[0, \infty)$. The solution is then given by (see old homework)

$$u(x,t) = \int_0^{+\infty} \frac{e^{-\frac{1}{2t}(x-y)^2} - e^{-\frac{1}{2t}(x+y)^2}}{\sqrt{2\pi t}} u_0(y)\, dy \tag{23.2}$$

Exercise 23.4 *Check that (23.2) is indeed a solution of the heat equation on $[0, \infty)$ with initial condition u_0. (Hint: use the "method of images".)*

In the homework you will learn about an application of the Fourier transform in the study of the wave equation.

23.2 The Poisson Summation Formula

The Poisson Summation Formula states that for a large class of functions $f : \mathbb{R} \to \mathbb{C}$ one has the followig intriguing identity

$$\sum_{n=-\infty}^{+\infty} f(2\pi n) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \hat{f}(n) \tag{23.3}$$

You can easily imagine that this formula can be useful to derive identies between infinite series, or even permit one to explicitly sum a series. In a slight variant, the Euler-MacLaurin formula, one can use it to find approcimate values for the sum of series with a an error term that can be easily estimated see, e.g., H. Dym and H.P. McKean, *Fourier series and integrals*, Academic Press, New York - London, 1972, p 112.). Note that the presence or absence of factors 2π in (23.3) depends on the chosen normalization for the Fourier transform.

Theorem 23.1 (Poisson Summation Formula) *Let $f \in C^1(\mathbb{R})$ such that there exists a constant $C > 0$, and $\epsilon > 0$, such that*

$$|(1+|x|)^{1+\epsilon} f(x)| \leq C, \quad |(1+|x|)^{1+\epsilon} f'(x)| \leq C, \quad \text{for all } x \in \mathbb{R}$$

Then one has the identity

$$\sum_{n=-\infty}^{+\infty} f(2\pi n) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \hat{f}(n)$$

Proof: First, note that, under the conditions of the theorem, the following sum

$$g(x) = \sum_{n=-\infty}^{+\infty} f(x + 2\pi n) \tag{23.4}$$

converges uniformly and defines a function $g \in C^1(S^1)$, i.e., a periodic continuously differentiable function on $\mathbb{R}$ with period 2π (see Exercise 23.5). Therefore, the Fourier series for g converges uniformly (Theorem 10.2):

$$g(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} e^{inx} \int_0^{2\pi} e^{-iny} g(y)\, dy$$

As g is related to f by (23.4), we can rewrite this as follows:

$$\sum_{n=-\infty}^{+\infty} f(x + 2\pi n) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} e^{inx} \hat{f}(n)$$

which for $x = 0$ reduces to the Poisson Summation Formula we set out to prove. ■

Exercise 23.5 *Suppose $(S_N)_{N=1}^{\infty}$ is a sequence of functions in $C^1(\mathbb{R})$ such that $S_N \to g$ and $S'_N \to h$, both uniformly. Prove that $g \in C^1(\mathbb{R})$, and $g' = h$.*

A standard application of the Poisson Summation Formula is an identity for the Jacobi theta function defined, for $t > 0$, by

$$\theta(t) = \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 t}$$

Jacobi's formula is then a symmetry property of this function expressed by

$$\theta(t) = \frac{1}{\sqrt{t}} \theta(1/t) \tag{23.5}$$

Exercise 23.6 *Prove (23.5).*

Exercise 23.7 *Prove the following identity for all $a > 0$:*

$$\sum_{n=-\infty}^{+\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \frac{1 + e^{-2\pi a}}{1 - e^{-2\pi a}}$$

Note that by considering the limit $a \downarrow 0$ of a rearrangement of this identity you can obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

which was a homework problem in MAT203B.

23.3 The Central Limit Theorem

Let X be a real random variable, with mean 0 and variance 1, and with distribution f. This means that f is a non-negative function in $L^1(\mathbb{R})$, satisfying

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1, \quad \int_{-\infty}^{+\infty} xf(x)\,dx = 0, \quad \int_{-\infty}^{+\infty} x^2 f(x)\,dx = 1 \tag{23.6}$$

The interpretation is that, for any $a \leq b$, the probability that X has a value between a and b is given by

$$\mathrm{Prob}\,(a \leq X \leq b) = \int_a^b f(x)\,dx$$

If X has a finite mean and variance, the normalization of the mean to 0 and the variance to 1 can be achieved by replacing X by Y defined by

$$Y = A + BX$$

with suitable real constants A and B (if this is not familiar to you, find A and B explicitly in terms of the mean and the variance of X). Therefore, the assumption that the mean vanishes and that the variance is 1 just amounts to assuming that these quantities are finite.

In study of random events described by such a random variable X, and in applications, it is often important to consider the sum of a large number of independent observations (or measurements) of X. Indeed, it is a standard procedure to reduce certain types of error in measurements of physical quantities in every experimental science, by measuring the quantity many times and taking the average. The central limit theorem explains how this error reduction works. In other applications the sum of many independent copies of X is the quantity of interest itself. E.g., a (discrete-time) *Random Walk* in one dimension is defined as the sequence of real numbers

$$(x_0, x_1, x_2, \ldots)$$

where $x_{n+1} = x_n + X_n$, i.e., at each time step the value of a random variable X_n is added to the previous position. E.g., we can take every X_n to be an independent copy of X. Independent means independent of the outcomes of X_m, for $m \neq n$. The random walker is then walking up and down the real line with steps that are independent of each other and that are chosen at random with distribution f. A natural question to ask is then what the position of the random walker is after N steps and assuming, e.g., that she started at 0.

For $N = 1$, the distribution of the position of the walker is obviously given by f. And if the position of the walker after N steps is x_N, the distribution of her position after $N + 1$ steps is clearly given by $f(x - x_N)$, because this is the probability that the $N+1$st step will be of magnitude $x - x_N$. Of course, x_N is random itself, say with distribution f_N. Then, the distribution at time step $N+1$ will be given by

$$f_{N+1}(x_{N+1}) = \int f(x_{N+1} - x_N) f_N(x_N)\, dx_N = f * f_N(x_{N+1})$$

It is then clear that the distribution of the position after N steps, starting from 0, i.e., $S_N = X_1 + X_2 + \cdots + X_N$, is given by

$$f_N = \underbrace{f * f * \cdots * f}_{N \text{ times}}$$

The Central Limit Theorem is a statement about the behavior of f_N in the limit of large N. As we should expect that typically values of S_N increase with N, the limit is better described by a properly normalized quantity. It turns out that the correct normalization is to consider

$$Y_N = \frac{1}{\sqrt{N}} S_N \tag{23.7}$$

Denote by g_N the distribution of Y_N. Then

$$g_N(x) = \sqrt{N} f_N(\sqrt{N} x)$$

The Central Theorem shows that, in the limit $N \to \infty$, g_N converges to a standard Gaussian distribution. The sense in which this convergence occurs, depends on the properties of f. Here we will prove that, for any $h \in S$,

$$\lim_{N\to\infty} \int g_N(x) h(x)\, dx = \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} h(x)\, dx$$

This means that the convergence is in the topology of $\mathcal{S}^*$. In probability theory this is called *weak convergence.*

By approximating the indicator function of the interval $[a, b]$ by test functions one can then also show that, for any $a < b$,

$$\lim_{N\to\infty} \text{Prob}\left(a \leq \frac{S_N}{\sqrt{N}} \leq b \right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}x^2}\, dx$$

Theorem 23.2 (Central Limit Theorem)

$$\lim_{N\to\infty} g_N(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \tag{23.8}$$

in the sense of distributions.

Proof: As the Fourier transform on $\mathcal{S}$ is a continuous transformation with continuous inverse, instead of (23.8), we can as well prove

$$\lim_{N\to\infty}\int g_N(k)\hat{h}(k) = \frac{1}{\sqrt{2\pi}}\int e^{-\frac{1}{2}k^2}\hat{h}(k)\,dk, \qquad \text{for all } h\in\mathcal{S}$$

Due to Plancherel's formula (22.9), and the formula for the Fourier transform of a Gaussian (22.3), this is then equivalent with

$$\lim_{N\to\infty}\int \widetilde{g_N}(k)h(k) = \frac{1}{\sqrt{2\pi}}\int e^{-\frac{1}{2}k^2}h(k)\,dk, \qquad \text{for all } h\in\mathcal{S} \tag{23.9}$$

As $g_N(x) = \sqrt{N}f_N(\sqrt{N}x)$, and f_N is the $N-$fold convolution of f with itself, we have that

$$\widehat{g_N}(k) = \widehat{f_N}(\frac{k}{\sqrt{N}})$$

As f_N is the $N-$fold convolution of f, we have, by Exercise 22.3,

$$\widehat{f_N}(\frac{k}{\sqrt{N}}) = (2\pi)^{(N-1)/2}(\hat{f}(\frac{k}{\sqrt{N}}))^N$$

The limit of the latter quantity can be most easily found by expanding e^{-iky} as follows

$$e^{-iky} = 1 - iky - \frac{1}{2}k^2y^2(1+r(ky))$$

where $r(y)$ is a continuous function of $y\in\mathbb{R}$, vanishing at $y=0$. By using the assumptions (23.6), we find the following expression for $\hat{f}(\frac{k}{\sqrt{N}})$:

$$\begin{aligned}\hat{f}(\frac{k}{\sqrt{N}}) &= \frac{1}{\sqrt{2\pi}}\int\left[1-\frac{ikx}{\sqrt{N}}-\frac{k^2x^2}{2N}(1+r(kx/\sqrt{N}))\right]f(x)\,dx\\ &= \frac{1}{\sqrt{2\pi}}\left[1-\frac{k^2}{2N}(1+R_N)\right]\end{aligned}$$

with $\lim_{N\to\infty}R_N = 0$ (by the dominated convergence theorem, see later). To compute the Nth power is now easy and we obtain

$$\lim_{N\to\infty}(2\pi)^{(N-1)/2}(\hat{f}(\frac{k}{\sqrt{N}}))^N = \lim_{N\to\infty}\frac{1}{\sqrt{2\pi}}\left[1-\frac{k^2}{2N}(1+R_N)\right]^N = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}k^2}$$

As the Fourier transform of the standard Gaussian distribution is the standard Gaussian distribution itself, we have proved that

$$g_n(x)\to\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$$

as a tempered distribution. ∎