Final Exam

(Take-home: June 12-19, due on Friday 06/19, by noon.)

Problem 1. Calculate the Fourier transforms of the following two tempered distributions.

a) For constants $a > 0$ and $\alpha < 0$:

$$\mathcal{S}(\mathbb{R}) \ni f \mapsto \phi_1(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\cos ak - e^{-\alpha}}{\cosh \alpha - \cos ka} f(k)\, dk$$

b) For constants $a > 0$, $b > 0$, and $\alpha < 0$:

$$\mathcal{S}(\mathbb{R}) \ni f \mapsto \phi_2(f) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1}{b^2 + k^2} \frac{\cos ak - e^{-\alpha}}{\cosh \alpha - \cos ka} f(k)\, dk$$

Problem 2. Consider the integro-differential equation of the form

$$-u''(x) + \lambda \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x-y)^2} u(y)\, dy = f(x)$$

where f and $g \in \mathcal{S}(\mathbb{R})$ are given. Prove that, for all $\lambda > 0$, this equation has a unique solution $u \in L^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$ and that for given g, this solution can be given in the form

$$u(x) = \int_{-\infty}^{+\infty} G(x-y) f(y)\, dy, \quad , \text{ for all } f \in \mathcal{S}(\mathbb{R}).$$

with $G \in C_0(\mathbb{R})$. Be sure to justify your arguments.

Problem 3. Define the following three sequences of functions $[0, \to +\infty) \to \mathbb{R}$:

$(f_n)_{n=1}^\infty$ given by

$$f_n(x) = \begin{cases} \frac{n^{1/2}}{(x+1)^n} & \text{if } 0 \le x \le n \\ 0 & \text{else} \end{cases} ,$$

$(g_n)_{n=1}^\infty$ given by

$$g_n(x) = \begin{cases} \sin(2\pi n x) & \text{if } n \le x \le n+1 \\ 0 & \text{else} \end{cases} ,$$

and $(h_n)_{n=1}^\infty$ given by

$$h_n(x) = \sum_{k=1}^{n} \frac{k}{\sqrt{n}} \mathrm{Ind}_{[k, k+(1/n^2)]}(x)$$

Consider these sequences with each of the topologies given below and determine whether or not they converge and, if they converge, determine their limits. Explain your assertions.

a) Pointwise on $[0, +\infty)$.

b) Uniformly on $[0, +\infty)$.

c) In the norm topology of $L^2([0, +\infty))$.

d) In the norm topology of $H^1([0, +\infty))$.

e) Strongly in $L^{3/2}([0, +\infty))$.

f) Weakly in $L^{3/2}([0, +\infty))$.

Problem 4. For any pair of real numbers α and β define the function $F_{\alpha,\beta} : \mathbb{R}^3 \to \mathbb{R}$, by

$$F_{\alpha,\beta}(x) = \|x\|^\alpha (1 + \|x\|)^\beta$$

Let $p \in [1, +\infty)$. For which pairs of α and β does the functional

$$\phi_{\alpha,\beta}(f) := \int F_{\alpha,\beta}(x) f(x)\, dx, \quad f \in \mathcal{S}(\mathbb{R}^3)$$

extend to a continuous linear functional on $L^p(\mathbb{R}^3, dx)$? Prove your answer.

Problem 5. Let Ω denote the unit square, i.e., $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$, and consider the Laplacian with Dirichlet boundary conditions on $L^2(\Omega)$, and call this operator Δ.

a) Compute the corresponding Green's function $g : \Omega \times \Omega \to \mathbb{R}$, as a Fourier series, i.e., find g such that for all $f \in L^2(\Omega)$, the function h defined by

$$h(x) = \int_\Omega g(x, y) f(y)\, dy$$

belongs to the domain of Δ and $\Delta h = f$.

b) Find a closed expression for g. (Hint : either sum the Fourier series for g, or use the one-dimensional case.)

Problem 6.

a) For $n = 1, 2, \ldots$, let $f_n : [0, +\infty) \to \mathbb{R}$ be continuous functions, and suppose there exists a constant $c \geq 0$ such that $f_n(x) \geq -c$, for all $x \geq 0$, and all $n \geq 1$. Prove that, for all $n \geq 1$, we have

$$\int_0^\infty e^{-f_n(x) - x}\, dx \geq e^{-\int_0^\infty e^{-x} f_n(x)\, dx} .$$

b) Suppose in addition that $f_n \to f$, pointwise a.e. Prove that the inequality of part a) also holds for f.