Assignment 1

(due date: Monday, April 13)

Problem 1. Let A be a linear operator on a Hilbert space $\mathcal{H}$ with dense domain $\mathcal{D}(A)$. Suppose $\mathcal{D}(A^*)$ is dense in $\mathcal{H}$. Prove that $A^{**} := (A^*)^*$, is an extension of A.

Problem 2. Let A and B be two linear operator on a Hilbert space $\mathcal{H}$ with domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$ respectively, and assume $\mathcal{D}(A) \cap \mathcal{D}(B)$ is dense. Define an operator C by $\mathcal{D}(C) = \mathcal{D}(A) \cap \mathcal{D}(B)$ and $Cx = Ax + Bx$, for all $x \in \mathcal{D}(C)$. Prove that C^* is an extension of $A^* + B^*$.

Problem 3. Let A and B be two linear operator on a Hilbert space $\mathcal{H}$ with domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$ respectively. Define $\mathcal{D}(AB)$ and $\mathcal{D}(B^*A^*)$ by

$$\begin{aligned} \mathcal{D}(AB) &= \{x \in \mathcal{D}(B) \mid Bx \in \mathcal{D}(A)\} \\ \mathcal{D}(B^*A^*) &= \{x \in \mathcal{D}(A^*) \mid A^*x \in \mathcal{D}(B^*)\} \end{aligned}$$

and assume that $\mathcal{D}(AB)$ is dense. Define operators AB and B^*A^* on their respective domains in the obvious way. Prove that $(AB)^*$ is an extension of B^*A^*.

Problem 4. Define a linear operator A on $L^2([0,1])$ as follows:

$$\begin{aligned} &\mathcal{D}(A) = \{f \in L^2([0,1]) \mid f \in H^1([0,1]), f(0) = f(1)\} \\ &Af = \frac{1}{i} f', f \in \mathcal{D}(A) \end{aligned}$$

Prove that A is self-adjoint. (Note: for now, defile $H^1([0,1])$ by

$$H^1([0,1]) = \{f \in L^2([0,1]) \mid f' \in L^2([0,1])\}.)$$