Assignment 2

(due date: Monday, April 20)

Problem 1. Consider the 2nd order differential operator on $L^2([0,1])$ given by

$$Lu = au'' + bu' + cu$$

with $a, b, c \in C([0,1])$, and $a(x) \neq 0$, for all $x \in [0,1]$.

a) Prove that for any pair of linearly independent solutions u_1, u_2 of $Lu = 0$, one has that their Wronskian is of the form

$$W_0 \exp\left[-\int_0^x \frac{b(y)}{a(y)}\, dy\right]$$

with W_0 a non-zero constant.

b) Prove that if L is Sturm-Liouville, i.e.,

$$Lu = -(pu')' + qu$$

with $p \in C^1([0,1]), q \in C([0,1])$, and p and q real, then the Green's function of L with Dirichlet b.c.'s can be written as

$$g(x,y) = \begin{cases} v_0(x)v_1(y) & \text{if } x < y \\ v_0(y)v_1(x) & \text{if } y < x \end{cases}$$

where $Lv_0 = 0, Lv_1 = 0, v_0(0) = 0 = v_1(1)$.

Problem 2. Let $\omega > 0$ be a constant, ω *not* integer, and consider the differential operator on $L^2([0,\pi])$ acting on C^2 functions as

$$Lu = u'' + \omega^2 u$$

a) Compute the Green's function g_D of L with *Dirichlet* b.c.

b) Compute the Green's function g_N of L with *Neumann* b.c. First convince yourself that all we said in class about the construction of the Green's function with Dirichlet b.c. carries over straightforwardly to the Neumann case.

c) Consider the integral operators G_D and G_N with integral kernels g_D and g_N repectively. We have

$$\begin{aligned} LG_D &= \mathbb{1} \text{ on } L^2([0,\pi]) \\ LG_N &= \mathbb{1} \text{ on } L^2([0,\pi]) \end{aligned}$$

So both are an inverse of L. Or are they not? How can they be different then?