Assignment 3

(due date: Monday, April 27)

Problem 1. Prove the following properties of the Fourier transform on test functions $f \in \mathcal{S}(\mathbb{R}^d)$:

a) [Behavior under translations] Recall the action of the translation over $a \in \mathbb{R}^d$ on $\mathcal{S}$: $T_a f(x) = f(x-a)$. Prove that

$$\widehat{T_a f}(k) = e^{-ik\cdot a}\hat{f}(k)$$

b) [Fourier transform of derivative] Prove that for every multi-index α

$$\widehat{\partial^\alpha f}(k) = (-ik)^\alpha \hat{f}(k)$$

c) [Fourier transform of convolutions] For all $f, g \in \mathcal{S}$

$$\frac{1}{(2\pi)^{d/2}}\widehat{f * g}(k) = \hat{f}(k)\hat{g}(k)$$

Problem 2. Let $k \in \mathcal{S}$ and define

$$Kf(x) = \int k(x-y)f(y)\,dy, \qquad \text{for all } f \in \mathcal{S}$$

Prove that $K : \mathcal{S} \to \mathcal{S}$, which is called a convolution operator, is a continuous linear operator for the topology of $\mathcal{S}$.

Problem 3. For every $a \in \mathbb{R}^n$ define a linear transformation $T_a : \mathcal{S} \to \mathcal{S}$ by $T_a(f)(x) = f(x-a)$. These are the translation operators on $\mathcal{S}$.

a) Prove that for all $a \in \mathbb{R}^d$, T_a is continuous for the topology of $\mathcal{S}$.

b) Prove that for all $f \in \mathcal{S}$, the map $a \mapsto T_a f$ is continuous from $\mathbb{R}^d$ to $\mathcal{S}$. (Hint: First, prove that for $f \in C(\mathbb{R}^n)$ one has $\lim_{a\to 0} \|T_a f - f\|_{\sup} = 0$ iff f is uniformly continuous. Also, note that it is sufficient to prove continuity in $a = 0$, due to the group property of T_a.)