Assignment 4

(due date: Monday, May 4)

Recall the wave equation in d dimensions (with speed of propagation set to one):

$$u_{tt} = u_{x_1x_1} + \cdots + u_{x_dx_d}$$

Suppose we would like to find the solution $u(x,t), t > 0, x \in \mathbb{R}^d$, of this equation, with initial data $u(x,0) = 0, u_t(x,0) = v_0(x)$. For simplicity assume $v_0 \in \mathcal{S}(\mathbb{R}^d)$. This could correspond, e.g., to the sound produced by an explosion near $x = 0$ at $t = 0$.

Problem 1. Let $\hat{u}$ denote the Fourier transform of u, i.e.,

$$\hat{u}(k,t) = \frac{1}{(2\pi)^{d/2}} \int e^{-ikx} u(x,t)\, dx$$

a) Find the equation satisfied by $\hat{u}$.

b) Show that u with the initial data given above is given by

$$u(x,t) = \frac{1}{(2\pi)^{d/2}} \int e^{ikx} \frac{\sin(\|k\|t)}{\|k\|} \hat{v}_0(k)\, dk$$

Problem 2. Suppose $d = 3$. Let $d\Omega_t$ denote the surface integration measure on the sphere of radius t, $t > 0$. So, $\int_{\|x\|=t} d\Omega_t = 4\pi t^2$.

a) Prove that the solution of Problem 1 can be written as

$$u(x,t) = \frac{1}{4\pi t} \int_{\|y\|=t} v_0(x+y) d\Omega_t \tag{1}$$

b) Consider v_0 with support = the ball of radius $\epsilon > 0$ centered at 0. Interprete this expression from the point of view of a person standing at position x and observing (hearing) the explosion that occurred at the origin.

Problem 3. Now consider $d = 2$. Prove that (1) has to be replaced by

$$u(x,t) = \frac{1}{2\pi} \int_{\|y\| \le t} \frac{v_0(x+y)}{\sqrt{t^2 - \|y\|^2}} dy$$

Interprete this formula and compare with the three-dimensional case. Hint: first, show that, for $d = 2$,

$$\frac{\sin(\|k\|t)}{\|k\|} = \frac{1}{2\pi} \int_{\|x\| \le t} \frac{e^{ikx}}{\sqrt{t^2 - \|x\|^2}}\, dx$$