Assignment 6

(due date: Wednesday, May 27)

Problem 1. Consider a measurable function $f : \Omega \times (0,1) \to \mathbb{R}$ with the following properties:
i) for all $t \in (0,1)$, $f(\cdot, t) \in L^1(\Omega, \mu)$;
ii) for all $x \in \Omega$, $f(x, \cdot)$ is differentiable on $(0,1)$;
iii) there exists $F \in L^1(\Omega, \mu)$ such that

$$\left| \frac{\partial}{\partial t} f(x,t) \right| \leq F(x),$$

for all $x \in \Omega$ and $t \in (0,1)$.
Using these properties of f, prove that differentiation and integration can be interchanged, i.e., prove that

$$g(t) := \int_\Omega f(x,t)\, \mu(dx)$$

is differentiable and that

$$g'(t) = \int_\Omega \frac{\partial}{\partial t} f(x,t)\, \mu(dx)$$

(Hint: use the dominated convergence theorem.)

Problem 2. Let $1 \leq p < +\infty$, and let $f_n : \Omega \to \mathbb{C}$ be a sequence of measurable functions converging to f pointwise μ-æ. Suppose there exists $g \in L^p(\Omega, \mu)$ such that $|f_n| \leq g$, μ-æ. Then $f_n \to f$ in the L^p-norm.

Problem 3. Prove the following generalization of Hölder's inequality: for n functions, $f_i \in L^{p_i}(\Omega, \mu)$, with $1 \leq p_i \leq +\infty$, such that $\sum_{i=1}^n 1/p_i = 1$, one has

$$\left| \int f_1(x) \cdots f_n(x)\, \mu(dx) \right| \leq \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}$$

Problem 4. Prove that for any pair of distinct exponents $1 \leq p, p' \leq +\infty$, we have $L^p(\mathbb{R}, dx) \not\subset L^{p'}(\mathbb{R}, dx)$.