Midterm

(Take-home: May 11-15, due Friday 05/15, 5 pm.)

Problem 1. Let $\mathcal{S}$ denote the following space of test functions:

$$\mathcal{S} = \left\{ f \in C^\infty([0,+\infty)) \,\middle|\, \begin{array}{l} (d/dx)^m f(0) = 0, \text{ for all } m \geq 0, \\ \text{and there exists } n \geq 0 \text{ such that } f(x) = 0, \text{for all } x \geq n \end{array} \right\}$$

and consider the differential operator L on $\mathcal{S}$ defined by:

$$Lu = u'' + 2u' + u$$

a) Find the solutions $u \in C^2([0,+\infty))$ of the differential equation

$$Lu = 0$$

b) Define a "Green's function" g by

$$g(x,y) = -\min(x,y)e^{y-x}$$

For $y \geq 0$ fixed, consider the distribution $G_y \in \mathcal{S}^*$ associated with g by

$$G_y(h) = \int_0^{+\infty} g(x,y)h(x)\,dx, \quad \text{for all } h \in \mathcal{S}$$

Compute the first and second derivatives of the distribution G_y.

c) Verify that, for each fixed $y \geq 0$, G_y satisfies

$$G_y'' + 2G_y' + G_y = \delta_y$$

where δ_y is the delta distribution at y.

d) Show that, for all $f \in \mathcal{S}$,

$$u(x) = \int_0^{+\infty} g(x,y)f(y)\,dy$$

satisfies $u'' + 2u' + u = f$.

Problem 2. Consider the sequence (ϕ_n) of tempered distributions on $\mathbb{R}$ defined by

$$\phi_n(f) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\sin nx}{x} f(x)\, dx$$

a) Prove that, for the topology of $\mathcal{S}^*(\mathbb{R})$, ϕ_n converges to the delta distribution at 0.

b) Compute the Fourier transforms $\hat{\phi}_n$.

c) Does the sequence $(\hat{\phi}_n)$ converge in $\mathcal{S}^*(\mathbb{R})$? If so, what is its limit?

Problem 3. Consider the integro-differential equation of the form

$$-u''(x) + \int_{-\infty}^{+\infty} g(x-y)u(y)\, dy = f(x)$$

where f and $g \in \mathcal{S}(\mathbb{R})$ are given, and suppose

$$\int_{-\infty}^{+\infty} g(x)\, dx > 0$$

Prove that this equation has a unique solution $u \in L^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$ and that for given g, this solution can be given in the form

$$u(x) = \int_{-\infty}^{+\infty} G(x-y)f(y)\, dy, \quad , \text{ for all } f \in \mathcal{S}(\mathbb{R}).$$

with $G \in C_0(\mathbb{R})$. Be sure to justify your arguments.

Problem 4. Let L be a (not necessarily bounded) operator on a Hilbert space $\mathcal{H}$ with dense domain $\mathcal{D}(L)$. Prove that for any $A \in \mathcal{B}(\mathcal{H})$, the operator $L + A$ with $\mathcal{D}(L+A) = \mathcal{D}(L)$, satisfies $(L+A)^* = L^* + A^*$ with domain $\mathcal{D}(L^* + A^*) = \mathcal{D}(L^*)$.