10 Special orthonormal bases 2. Wavelets

In this chapter we will introduce a special class of o.n. bases of $L^2([0,T])$ and $L^2(\mathbb{R})$, called *wavelet* bases. These bases have proved to be very useful in signal analysis, image processing, and data compression. For more information on the mathematics and applications of wavelets, see one or more of the following references:

- For an an elementary introduction, see
  J. Walker, *Fourier Analysis and Wavelet Analysis*, Notices of the AMS, **44** (1997) 658–670, and references therein.

- More examples are in
  G. Strang, *Wavelets and dilation equations: a brief introduction*, SIAM Review, **31** (1989), 614–627.

- This paper has more of the math:
  S. Mallat, *Multiresolution approximations and wavelet orthonormal basis of $L^2(\mathbb{R})$.*, Trans. American Math. Soc., **315** (1989), 69–87.

- Read this if you are interested in applications to data compression, image coding etc.
  S. Mallat, *A theory of multiresolution sigma decomposition: the wavelet approximation*, IEEE Trans. Pattern Anal. Machine Intell., **11** (1989), 674–693.

- For a more complete account see the book
  I. Daubechies, *Ten lectures on wavelets*, SIAM, 1992.

- Some routines for numerical implementation are described in
  W.H. Press et al., *Numerical Recipes in C*, Cambridge University Press, Second Edition, 1992

10.1 Fourier series versus wavelets

Fourier series expansions are very effective for representing "nice" periodic functions. For other types of functions, such as functions that vanish on most of their domain, they are not very efficient in the sense that a partial sum with a large number of terms is needed to obtain an good approximation of the function. In signal processing such functions are called *transients*. Recall Parseval's identity, Theorem 7.11 iii):

$$||f||^2 = \sum_n |\hat{f}(n)|^2$$

From this relation we should expect that a large number of terms are needed for a good approximation if the quantity $||f||$ is distributed over a large number of not-so-small coefficients $\hat{f}(n)$. By Theorem 8.2, this must happen, e.g., when f is discontinuous or non-smooth.

Transient signals, supported on a relatively small portion of the relevant time interval, and signals with sharp –almost discontinuous–, transitions are very common. A digitized signal, e.g., may consist of a series of irregularly spaced voltage spikes, or short beeps transmitted by a modem.

Often it is useful to compress a signal before transmission or storage. E.g., to accurately represent a function $f(t), t \in [0, T]$, by storing a finite number of values $f(k\Delta t), k = 0, \ldots, N, N = T/\Delta t$, one needs to choose Δt small enough such that all rapid transitions (or short beeps) can be reconstructed from this list of values. If changes are rapid, Δt will have to be small, and, therefore N will be large. Suppose, however, that we have an o.n. basis of $L^2([0, T])$, with the property that a finite linear combination of basis elements, say with N_0 terms, $N_0 \ll N$, yields a good approximation of the function f. In order to store or transmit the function we can then suffice with N_0 instead of N numbers (the coefficients in the linear combination), without giving up much of the interesting information. Roughly speaking, we have achieved a compression of the data with compression ratio $N : N_0$.

A certain class of o.n. bases, called wavelets, have become very popular for representing signals (functions of time) and images (functions of $x \in I \subset \mathbb{R}^2$) during the last decade, because they allow for large compression ratios. There are different kinds of wavelets, but all share the property that they describe the function at a sequence of different time or length scales. Before we explain precisely what this means, it is instructive to first consider a simple example.

10.2 The Haar wavelets

Consider the Hilbert space $L^2([0, 1])$ with its usual inner product. Later we will also consider $L^2(\mathbb{R})$. Define

$$C_n = \{f \in L^2([0, 1]) \mid f \text{ is constant on } [k/2^n, (k+1)/2^n), \text{ for all } k = 0, 1, \ldots, 2^n - 1\} \tag{10.1}$$

Note that for all $n \geq 0$, C_n is a finite-dimensional, and hence closed (Theorem 6.13), linear subspace of $L^2([0, 1])$. Clearly $C_n \subset C_{n+1}$, for all $n \geq 0$. Define $\psi \in L^2([0, 1])$ by

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1/2 \\ -1 & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

Let $B_0 = \{1\}$, where 1 denotes the constant function $= 1$. For $n \geq 0$ define

$$B_{n+1} = \{x \mapsto 2^{n/2}\psi(2^n x - k) \mid k = 0, 1, \ldots, 2^n - 1\} \tag{10.2}$$

Note that $B_n \subset C_n$.

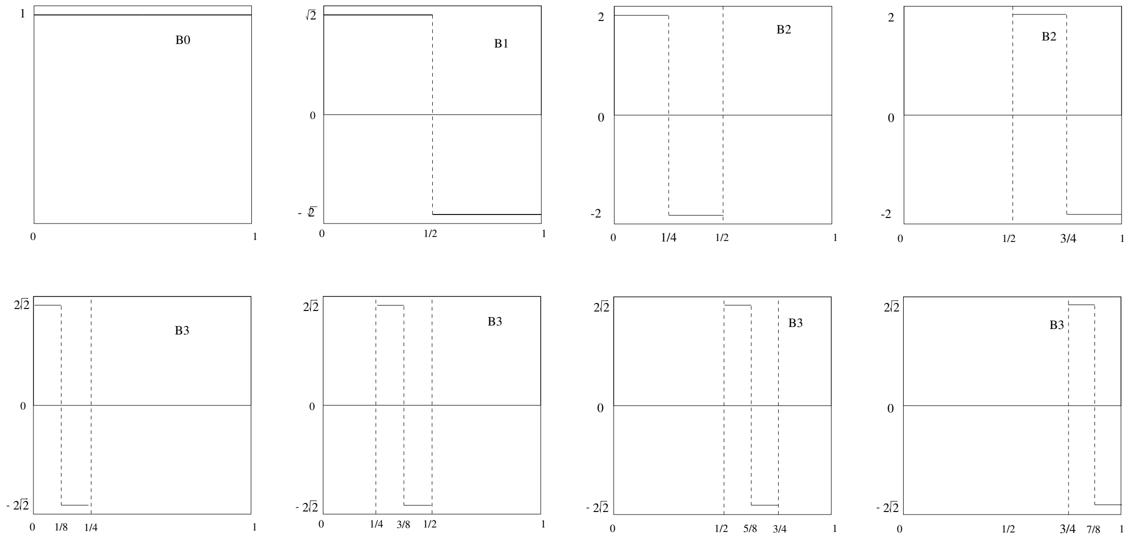

Figure 10.1: Some members of the sets of functions B_n defined in (10.2).

Exercise 10.1 *Prove that $\cup_{n=0}^{N} B_n$ is an o.n. basis of C_N. (Hint: prove that the set is orthonormal and count its elements.)*

Lemma 10.1 *$\mathcal{B} := \cup_{n=0}^{\infty} B_n$ is an o.n. basis of $L^2([0,1])$*

Proof: From the previous exercise we already know that $\mathcal{B}$ is an o.n. set. As the continuous functions are dense in $L^2([0,1])$, it is sufficient to show that for every $f \in C([0,1])$, and $\epsilon > 0$, there is a finite linear combination of elements in $\mathcal{B}$, i.e., an element g in one of the spaces C_n, such that $\|f-g\|_{L^2} \leq \epsilon$. Actually we will prove the stronger statement that there is $g \in C_N$, for a suitable N, such that $\|f-g\|_{\sup} \leq \epsilon$.

By the uniform continuity of f (Theorem 2.11), there exists N such that for all $x, y \in [0,1]$, with $|x-y| \leq 2^{-N}$, one has $|f(x)-f(y)| \leq \epsilon$. It follows that $g \in C_N$ defined by

$$g(x) = \sum_{k=0}^{2^N-1} f(k/2^N)\mathrm{Ind}_{[k/2^N,(k+1)/2^N)}(x)$$

satisfies the desired bound. ■

Exercise 10.2 *Suppose $\mathcal{B} = \{e_n\}_{n=1}^{\infty}$ is an o.n. basis for $L^2([0,1])$. Prove the following:*

a) *For any $a \in \mathbb{R}$, $\mathcal{B}_a = \{x \mapsto e_n(x-a)\}_{n=1}^{\infty}$ is an o.n. basis for $L^2([a,a+1])$.*

b) *For any $c > 0$, $\mathcal{B}^c = \{x \mapsto \sqrt{c}e_n(cx)\}_{n=1}^{\infty}$ is an o.n. basis for $L^2([0,c^{-1}])$.*

c) *With the convention that functions are extended to a larger than their original domain by putting them equal to 0, prove that $\mathcal{B} \cup \mathcal{B}_1$ is an o.n. basis for $L^2([0,2])$.*

d) *Prove that $\cup_{k\in\mathbb{Z}}\mathcal{B}_k$ is an o.n. basis for $L^2(\mathbb{R})$.*

The basis $\mathcal{B} = \cup_{k\in\mathbb{Z}}\mathcal{B}_k$ is called the *Haar wavelet basis* of $L^2(\mathbb{R})$, and its elements are called the *Haar wavelets* . The Haar wavelets are very simple functions, step-functions each taking only three different values. They have compact support and have a simple scaling behavior, as illustrated in the following exercise.

Exercise 10.3 *With $\mathcal{B}$ as in Lemma 10.1 and with the notations and conventions of the previous exercise, show that $\mathcal{B}^{1/2}$ considered as an o.n. basis of $L^2([0,2])$ is equal to $B \cup B_1$.*

These properties, compact support and scaling, make the Haar wavelets especially suitable for decomposition of functions with a compact support that is small compared to their domain. But as the Haar wavelets are not continuous, partial sums approximating a continuous function are not continuous. It is often desirable to have continuous approximations of continuous functions, or C^p approximations of C^p functions. This is one motivation for introducing other wavelet basis.

10.3 Multiresolution analysis

In this section we want to illustrate how other wavelet bases can be constructed. We will mention the Daub4 wavelet basis, due to Ingrid Daubechies (1988).

We will consider wavelet bases for $L^2(\mathbb{R})$. Multiresolution analysis refers to the following structure. There is a family of closed linear subspaces of $L^2(\mathbb{R})$, indexed by $n \in \mathbb{Z}$, and denoted by $\{C_n\}$, and a function $\varphi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ such that the following properties hold:

MRA1: Scaling. $x \mapsto f(x) \in C_n$ iff $x \mapsto f(2x) \in C_{n+1}$, for all $n \in \mathbb{Z}$.

MRA2: Inclusion. $C_n \subset C_{n+1}$, for all $n \in \mathbb{Z}$.

MRA3: Density. $\overline{\bigcup_{n\in\mathbb{Z}} C_n} = L^2(\mathbb{R})$.

MRA4: Maximality. $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$.

MRA5: Basis. $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ is an o.n. basis of C_0.

These properties are the "axioms of multiresolution analysis". Note that the spaces C_n defined in (10.1) (extend the definition to all integers n), and the function $\varphi(x) = \mathrm{Ind}_{[0,1)}(x)$, satisfy MRA1-5.

Next, we show how a wavelet basis can be obtained from this structure. For the special case described in the previous paragraph the procedure will reproduce the Haar wavelets. But, as we will explain, there are other possibilities. In particular, there we will see how to construct φ that lead to an orthonormal wavelet basis consisting of continuous functions with compact support.

From MRA1 and MRA5 it follows that

$$B_n := \{2^{n/2}\varphi(2^n x - k) \mid k \in \mathbb{Z}\}$$

is an o.n. basis of C_n, for all $n \in \mathbb{Z}$. Of course, $\bigcup_n B_n$ is not a linearly independent set, due to MRA2. For this reason, we define closed linear subspaces of $L^2(\mathbb{R})$ by

$$C_{n+1} = C_n \oplus W_n \tag{10.3}$$

The W_n are called the wavelet subspaces and using MRA2-4, one can easily prove that

$$\bigoplus_{n\in\mathbb{Z}} W_n := \overline{\bigcup_{N\geq 1} \bigoplus_{-N\leq n\leq N} W_n} = L^2(\mathbb{R}) \tag{10.4}$$

Recall that in our discussion of the Haar basis we used the function ψ given by

$$\psi(x) = \varphi(2x) - \varphi(2x-1) \tag{10.5}$$

And one also has

$$\varphi(x) = \varphi(2x) + \varphi(2x-1) \tag{10.6}$$

We will generalize this relation between φ and its rescaled version and between φ and ψ later. For now, just observe the following properties: $x \mapsto \psi(x-k) \in C_1$ and $\perp C_0$; hence, these functions are in W_0. We already know that $\{\varphi(2x-2k) - \varphi(2x-(2k+1)) \mid k \in \mathbb{Z}\} \cup \{\varphi(2x-2k) + \varphi(2x-(2k+1)) \mid k \in \mathbb{Z}\}$ is an o.n. basis of C_1. From (10.6) we then see that $\{\varphi(2x-2k) + \varphi(2x-(2k+1)) \mid k \in \mathbb{Z}\}$ is an o.n. basis of C_0. From this and (10.5) it follows that $\{x \mapsto 2^{n/2}\psi(2^n x - k) \mid k \in \mathbb{Z}\}$ is an o.n. basis of W_0. By combining this with MRA1 we obtain the following lemma.

Lemma 10.2
i) $\{x \mapsto 2^{n/2}\psi(2^n x - k) \mid k \in \mathbb{Z}\}$ *is an o.n. basis for* W_n.
ii) $\{x \mapsto 2^{n/2}\psi(2^n x - k) \mid n, k \in \mathbb{Z}\}$ *is an o.n. basis for* $L^2(\mathbb{R})$.

So far, we have just rederived the Haar wavelet basis. The spaces W_n are also useful for investigating the practical issues involved with the Fast Haar Transform, which is important for numerical applications. See, e.g., the paper by Walker for a discussion.

Next, we explain how the same structure leads to other wavelet bases, in particular the continuous Daub4 wavelets. To this end we will make some further assumptions about the functions φ and ψ (which are satisfied in the Haar case).

Suppose we have real-valued functions φ and ψ with the following properties:

1. Normalization properties

$$\int_{\mathbb{R}} \varphi(x)\, dx = 1 \int_{\mathbb{R}} \varphi(x)^2\, dx = 1, \int_{\mathbb{R}} \psi(x)^2\, dx = 1 \quad .$$

2. The functions $\{x \mapsto \varphi(x-k) \mid k \in \mathbb{Z}\}$ are an orthonormal set.

3. Orthogonality of φ and ψ:

$$\int_{\mathbb{R}} \varphi(x)\psi(x)\, dx = 0$$

4. l vanishing moments of ψ:

$$\int_{\mathbb{R}} x^j \psi(x)\, dx = 0, \quad 0 \le j \le l-1$$

Define

$$C_0 = [\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}]$$

where, as before $[\cdot]$ denotes the closed linear span. Suppose that $\{C_n\}_{n\in\mathbb{Z}}$ and φ satisfy MRA1-5. Then, by MRA1,

$$C_n = [\{x \mapsto 2^{n/2}\varphi(2^n x - k)\}_{k\in\mathbb{Z}}\}]$$

From MRA2 we see that $\varphi \in C_1$, and, therefore, φ can be expanded in the basis of rescaled and shifted functions as follows

$$\varphi(x) = \sum_{k\in\mathbb{Z}} a_k \varphi(2x-k) \tag{10.7}$$

In order to facilitate finding examples of useful pairs of functions φ and ψ, we propose the following form for ψ:

$$\varphi(x) = \sum_{k\in\mathbb{Z}} (-1)^k a_{1-k} \varphi(2x-k) \tag{10.8}$$

From the orthogonality property 3 we then derive the following equation for the coefficients a_k:

$$\sum_{k\in\mathbb{Z}} (-1)^k a_{1-k} a_k = 0$$

The normalization properties 1) translate into

$$\sum_{k\in\mathbb{Z}} a_k = 2; \quad \sum_{k\in\mathbb{Z}} a_k^2 = 2$$

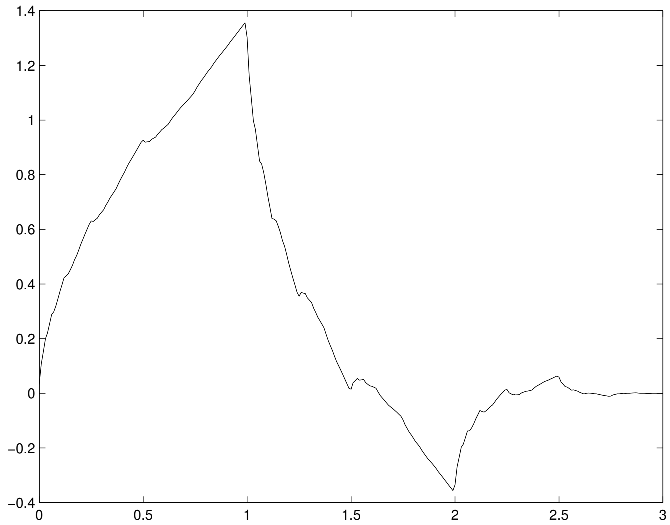

Figure 10.2: The function ψ for the Daub4 wavelets.

Note that the Haar case is a solution with $l = 1, \varphi = \mathrm{Ind}_{[0,1)}$, and $a_0 = a_1 = 1, a_k = 0$, for $k \neq 0, 1$. Daubechies's solution with $l - 2$ is determined by

$$\begin{aligned} a_0 &= \frac{1}{4}(1+\sqrt{3}) \quad , \quad a_1 = \frac{1}{4}(3+\sqrt{3}) \\ a_2 &= \frac{1}{4}(3-\sqrt{3}) \quad , \quad a_3 = \frac{1}{4}(1-\sqrt{3}) \\ a_k &= 0 \qquad \text{otherwise} \end{aligned}$$

The corresponding φ is determined by the equation (10.7), with the specified values for the coefficients a_k. One can find φ by solving (10.7) iteratively (regard it as a fixed point equation):

$$\begin{aligned} \varphi_{n+1}(x) &= \sum_{k\in\mathbb{Z}} a_k \varphi_n(2x-k), n \geq 0 \\ \varphi_0(x) &= \mathrm{Ind}_{[0,1)}(x) \end{aligned}$$

In fact, (10.7) is an eigenvalue equation with eigenvalue 1 and a rather complicated linear operator which is combination of translations of the rescaled function. One can show that φ_n converges to a continuous function with support in $[0, 3]$. Clearly, ψ, defined by (10.8), is then also continuous and with support contained in $[0, 3]$. These functions are not differentiable however. Their derivatives have an infinite number of discontinuities, as can be guessed from their graphs (see Figure 10.2).