# 11 Linear operators on a Hilbert space

Here, we introduce some special classes of bounded linear operators on a Hilbert space $\mathcal{H}$ or from a Hilbert space $\mathcal{H}_1$ to another Hilbert space $\mathcal{H}_2$.

## 11.1 Unitary operators

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. A linear map $U : \mathcal{H}_1 \to \mathcal{H}_2$ is called *unitary* if it is invertible and

$$\langle Ux, Uy \rangle_{\mathcal{H}_2} = \langle x, y \rangle_{\mathcal{H}_1},$$

i.e., $U$ is one-to-one and preserves the inner product. Two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are said to be *isomorphic* (as Hilbert spaces) if there exists a unitary linear map between them.

**Exercise 11.1 a)** *Show that a linear operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ is unitary iff it is an isometric isomorphism of normed linear spaces.*
**b)** *Show that an invertible linear map is unitary iff its inverse is.*

**Examples:**

1. Let $\{u_\alpha\}$ be an o.n. basis of $\mathcal{H}_1$ and $\{v_\alpha\}$ an o.n. basis of $\mathcal{H}_2$. Define $U : \mathcal{H}_1 \to \mathcal{H}_2$ as follows. Any $x \in \mathcal{H}_1$ can be written uniquely as

$$x = \sum_\alpha c_\alpha u_\alpha$$

with coefficients $c_\alpha \in \mathbb{C}$. Define $Ux$ by

$$Ux = \sum_\alpha c_\alpha v_\alpha$$

Then $U$ is a linear unitary map.

2. Let $\{u_\alpha\}$ be an o.n. basis of $\mathcal{H}$ and for all $\alpha$ pick $\lambda_\alpha \in \mathbb{C}$ with $|\lambda_\alpha| = 1$. Then

$$Ux = \sum_\alpha \lambda_\alpha \langle u_\alpha, x \rangle u_\alpha$$

is unitary.

3. Consider $\mathcal{H}_1 = L^2(T)$ and $\mathcal{H}_2 = l^2(\mathbb{Z})$. Then

$$Uf = (a_n)$$

where $(a_n)$ is the sequence of Fourier coefficients of $f$, i.e.,

$$a_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} f(x)\, dx,$$

is unitary.

4. For $a \in T$ and $f \in L^2(T)$ define

$$U_a f(x) = f(x - a)$$

Then $U_a$ ia unitary. The set of all such $U_a$ form a representation of the additive group $\mathbb{R}/(2\pi)\mathbb{Z}$:

$$U_{a+b} = U_a U_b$$

5. Let $\mathcal{H} = \mathbb{C}^n$. A complex $n \times n$ matrix represents a unitary linear map $\mathcal{H} \to \mathcal{H}$ iff $U^* U = \mathbb{1}$, where $U^*$ is the hermitian conjugate of $U$.

6. Let $\mathcal{H} = \mathbb{C}^n$. For any hermitian $n \times n$ matrix $A$ the matrix $U = \exp(iA)$ represents a unitary linear map $\mathcal{H} \to \mathcal{H}$, where, for any matrix $B$

$$\exp B = \sum_{n=0}^{\infty} \frac{1}{n!} B^n$$

**Exercise 11.2** *Let $U : \mathcal{H}_1 \to \mathcal{H}_2$ be a unitary linear map, and $\{u_\alpha\}$ an o.n. basis of $\mathcal{H}_1$. Show that $\{v_\alpha = U u_\alpha\}$ is an o.n. basis of $\mathcal{H}_2$.*

## 11.2 Orthogonal Projections

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}_1$ and $\mathcal{H}_2$ be two subspaces of $\mathcal{H}$. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are mutually orthogonal and such that for all $x \in \mathcal{H}$, there are unique vectors $x_1 \in \mathcal{H}_1$ and $x_2 \in \mathcal{H}_2$, such that $x = x_1 + x_2$, then we say that $\mathcal{H}$ is the direct sum of $\mathcal{H}_1$ and $\mathcal{H}_2$. The standard notation for this situation is

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \tag{11.1}$$

and one writes $x = x_1 \oplus x_2$, where $x_1$ and $x_2$ are the unique vectors in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, for which $x = x_1 + x_2$. It follows that $\mathcal{H}_1^\perp = \mathcal{H}_2$ and $\mathcal{H}_2^\perp = \mathcal{H}_1$, and hence, by Theorem 1.5, $\mathcal{H}_1$ and $\mathcal{H}_2$ are closed. (11.1) is called an *orthogonal decomposition* of $\mathcal{H}$. Similarly, for any two closed and mutually orthogonal subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of a Hilbert space $\mathcal{H}$, one has that

$$\mathcal{H}_1 + \mathcal{H}_2 = \mathcal{H}_1 \oplus \mathcal{H}_2$$

is a closed linear subspace of $\mathcal{H}$. The same notation is used for the construction that defines a new Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$ for any two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. $\mathcal{H}_1 \oplus \mathcal{H}_2$ is then called the *orthogonal sum* or *direct sum* of $\mathcal{H}_1$ and $\mathcal{H}_2$.

For every closed linear subspace $\mathcal{H}_1$ of $\mathcal{H}$ there is an orthogonal decomposition of $\mathcal{H}$. It suffices to take $\mathcal{H}_2 = \mathcal{H}_1^{\perp}$. This is a consequence of Theorem 7.6.

A linear map $P : \mathcal{H} \to \mathcal{H}$, satisfying $P^2 = P$ and $\langle Px, y\rangle = \langle x, Py\rangle$, is called an *orthogonal projection* on $\mathcal{H}$..

Next, we show that orthogonal projections $P$ are bounded, and that $\|P\| = 1$ except for $P = 0$. If $P \neq 0$, there is $x \in \mathcal{H}$ such that $Px \neq 0$, and we have

$$\begin{aligned} \|Px\| &= \langle Px, Px\rangle/\|Px\| = \langle x, P^2x/\|Px\|\rangle \\ &= \langle x, Px/\|Px\|\rangle \leq \|x\| \left\| \frac{Px}{\|Px\|} \right\| = \|x\| \end{aligned}$$

Therefore, we always have $\|P\| \leq 1$. In order to prove the other inequality we start from $\|P(Px)\| = \|Px\|$. Therefore, $\|P\| \geq 1$. Hence, $\|P\| = 1$, if $P \neq 0$.

For a linear map $A : \mathcal{H} \to \mathcal{H}$ the *range* of $A$, $\mathrm{ran}A$, is defined by

$$\mathrm{ran}A = \{y \in \mathcal{H} \mid \text{ there exists } x \in \mathcal{H} \text{ such that } y = Ax\}$$

**Exercise 11.3** *Prove that if $P$ is an orthogonal projection, then $x \in \mathrm{ran}P$ iff $Px = x$. (Hint: use $P^2 = P$).*

**Exercise 11.4** *Check that if $P$ is an orthogonal projection, $\mathrm{ran}P = (\ker P)^{\perp}$. (Hint prove $\mathrm{ran}P \subset (\ker P)^{\perp}$ and $\ker P \subset (\mathrm{ran}P)^{\perp}$.)*

**Lemma 11.1** *If $P$ is an orthogonal projection on $\mathcal{H}$, then $\mathrm{ran}P$ is a closed linear subspace of $\mathcal{H}$.*

**Proof:** That $\mathrm{ran}P$ is a linear subspace follows from the linearity of $P$. That $\mathrm{ran}P$ is closed follows from Exercise 11.4 and Theorem 7.5. ■

Next, we show that there is a one-to-one correspondence between orthogonal projections and closed linear subspaces, or, equivalently, between orthogonal projections and orthogonal decompositions of $\mathcal{H}$.

**Theorem 11.2** *i) Let $P$ be any orthogonal projection on $\mathcal{H}$. Then*

$$\mathcal{H} = \mathrm{ran}P \oplus \ker P$$

*ii) If $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, then*

$$Px = x_1, \quad \textit{for } x = x_1 \oplus x_2$$

*defines on orthogonal projection on $\mathcal{H}$ with $\mathrm{ran}P = \mathcal{H}_1$ and $\ker P = \mathcal{H}_2$.*

**Proof:**
*Proof of i):* This is a direct consequence from the fact that $\ker P$ is a closed linear subspace and Exercise 11.4.
*Proof of ii):* The only non-trivial property to check is that $P$ satisfies $\langle Px, y\rangle = \langle x, Py\rangle$, for all $x, y \in \mathcal{H}$. This follows from the linearity of the inner product and the orthogonality of the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. ∎

**Examples:**

1. One-dimensional projections. For any vector $u \in \mathcal{H}$, with $\|u\| = 1$, $P_u$ defined by
$$P_u x = \langle u, x\rangle u$$
is an orthogonal projection. E.g., let $\mathcal{H} = l^2(\mathbb{Z})$, and $u = e_n$, where for any $n \in \mathbb{Z}$
$$e_n = (\delta_{k,n})_{k=-\infty}^{\infty}$$
Then, for $x = (x_k)$,
$$P_{e_n} x = x_n e_n$$
Physicists like to denote $P_u$ by $|u\rangle\langle u|$.

2. Let $\mathcal{H} = L^2(\mathbb{R})$ and $[a, b] \subset \mathbb{R}$. Then
$$P_{[a,b]} f(x) = \chi_{[a,b]}(x) f(x)$$
is an orthogonal projection. Here $\chi_{[a,b]}$ is the *characteristic function* or *indicator function* of the interval $[a, b]$. For any subset $A$ of a set $X$, $\chi_A : X \to \{0, 1\}$ defined by
$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

3. If $P$ is an orthogonal projection on $\mathcal{H}$, then so is $Q = \mathbb{1} - P$. Note that $\ker Q = \mathrm{ran} P$ and $\mathrm{ran} Q = \ker P$.

4. Let $\mathcal{H} = L^2(T)$. Then
$$Pf = f - \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx$$
defines an orthogonal projection on $\mathcal{H}$.

5. If $U$ is unitary and such that $U^2 = 1$, then $P = (\mathbb{1} + U)/2$, is an orthogonal projection.

**Exercise 11.5** *Let $\{u_\alpha\}$ be an orthogonal set of vectors in a Hilbert space $\mathcal{H}$. Prove that $\{u_\alpha\}$ is an o.n. basis iff*

$$\sum_\alpha |u_\alpha\rangle\langle u_\alpha| = \mathbb{1}$$

One way in which general Banach spaces differ from Hilbert spaces is that there are, in general, not enough continuous projections. See Section 50 in Simmons for a discussion.