12 Linear functionals on a Hilbert space

12.1 The dual of a Hilbert space

A *linear functional* on a complex Hilbert space $\mathcal{H}$ is a linear map $\mathcal{H} \to \mathbb{C}$. As linear functionals are a special case of linear maps between Banach spaces, we have that a linear functional φ is continuous iff it is bounded, i.e., iff there exists a constant M such that

$$|\varphi(x)| \leq M\|x\|, \quad \text{for all } x \in \mathcal{H} \tag{12.1}$$

The smallest constant for which this inequality holds is then the norm of the functional φ, i.e.,

$$\|\varphi\| = \sup\{|\varphi(x)| \mid x \in \mathcal{H}, \|x\| = 1\} \tag{12.2}$$

Example:
For any $y \in \mathcal{H}$,

$$\varphi_y(x) = \langle y, x\rangle \tag{12.3}$$

defines a bounded linear functional on $\mathcal{H}$ with norm $\|\varphi_y\| = \|y\|$. This equality for the norm is a consequence of the following lemma.

Lemma 12.1 *For any $x \in \mathcal{H}$ one has*

$$\|x\| = \sup\{|\langle x, y\rangle| \mid y \in \mathcal{H}, \|y\| = 1\} \tag{12.4}$$

Proof: For $x = 0$ the equality is obvious. So suppose $x \neq 0$. By taking $y = x/\|x\|$, we see that $\|x\| \leq \sup\{\cdots\}$. On the other hand the inner product satisfies $|\langle x, y\rangle| \leq \|x\|\|y\|$. Hence, by taking $\|y\| = 1$, we also have $\sup\{\cdots\} \leq \|x\|$. This proves (12.4). ∎

Example:
Let $\mathcal{H} = L^2([0,\pi])$. Then, for each $n = 1, 2, \ldots$ the maps a_n defined by

$$a_n(f) = \sqrt{2/\pi}\int_0^\pi \sin(nx)f(x)\,dx$$

that associate to any function its n^{th} Fourier coefficient, are continuous linear functionals on $L^2([0,\pi])$.

The space of all bounded ($\equiv$ continuous) linear functionals on a Banach space X (e.g., a Hilbert space), X^*, is again a Banach space with the linear structure determined by

$$(\varphi + \lambda\xi)(x) = \varphi(x) + \lambda\xi(x), \quad \text{for all } \varphi, \eta \in X^*, \lambda \in \mathbb{C}, x \in \mathcal{H}$$

and the norm defined by (12.2) (see Theorem 6.6). X^* is called the *dual* (space) of X.

The following theorem tells us that the examples of bounded linear functionals on a Hilbert space determined by a vector $y \in \mathcal{H}$ as in (12.3), are in fact *all* bounded linear functionals on $\mathcal{H}$.

Theorem 12.2 (The Riesz Representation Theorem) *Let φ be a bounded linear functional on a Hilbert space $\mathcal{H}$, i.e., $\varphi \in \mathcal{H}^*$. Then there is a unique vector $y \in \mathcal{H}$ such that*

$$\varphi(x) = \langle y, x \rangle \tag{12.5}$$

Proof: We already know that $\varphi_y(x) = \langle y, x\rangle$ defines a bounded linear functional on $\mathcal{H}$. It is also easy to see that if $y_1 \neq y_2$, then $\varphi_{y_1} \neq \varphi_{y_2}$. Indeed, if $\varphi_{y_1}(x) = \varphi_{y_2}(x)$, for all $x \in \mathcal{H}$, then $\langle y_1 - y_2, x\rangle = 0$, for all x, and in particular for $x = y_1 - y_2$. This implies $\|y_1 - y_2\| = 0$, and hence, $y_1 = y_2$.

The non-trivial part of the proof of this theorem is to show that any functional φ is of the form (12.5) for some $y \in \mathcal{H}$. If $\varphi = 0$, we can take $y = 0$. So, suppose $\varphi \neq 0$. Then there exists $z \in \mathcal{H}$ such that $z \perp \ker\varphi$. Pick such a vector z. We claim that there is a constant $\lambda \in \mathbb{C}$ such that

$$\varphi(x) = \langle \lambda z, x\rangle$$

λ is then necessarily given by

$$\lambda = \frac{\overline{\varphi(z)}}{\|z\|^2} \quad .$$

In order to prove this consider the linear map P defined by

$$Px = \frac{\varphi(x)}{\varphi(z)}\, z$$

It is then straightforward to check that P is linear, and that $P^2 = P$. Then $\mathrm{ran}P = \{\alpha z \mid \alpha \in \mathbb{C}\}$, and $\ker P = \ker\varphi$, and in particular $\mathrm{ran}P \perp \ker P$, from which it follows that P is an orthogonal projection ($x = Px + (x - Px) \Rightarrow \mathcal{H} = \mathrm{ran}P \oplus \ker P$, and from this it is easy to check that $\langle Px, y\rangle = \langle x, Py\rangle$, for all $x, y \in \mathcal{H}$). So, we have proved that

$$\mathcal{H} = \ker\varphi \oplus \{\alpha z \mid \alpha \in \mathbb{C}\}$$

and for any $x = x_1 + \alpha z$, with $x_1 \in \ker\varphi$ we have

$$\varphi(x) = \alpha\varphi(z) = \langle \frac{\overline{\varphi(z)}}{\|z\|^2} z, z\rangle = \langle \lambda z, x\rangle$$

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One says that Hilbert spaces are *selfdual*: $\mathcal{H}^*$ is (anti)isomorphic to $\mathcal{H}$; $\mathcal{H}$ and $\mathcal{H}^*$ are isomorphic as Banach spaces, and anti-isomorphic as Hilbert spaces: the map $y \mapsto \varphi_y$ is one-to-one and onto, and it preserves the norm, but it is anti-linear: $\varphi_{\lambda y} = \overline{\lambda}\varphi_y$.

Later we will consider the duals of other Banach spaces (e.g., L^p-spaces, and $C([a,b])$), and we will see that a Banach space is in general quite different from its dual. Hilbert spaces are special in this respect.

Exercise 12.1 *Prove that $\mathcal{H}^*$ is a Hilbert space with the inner product defined by*

$$\langle \varphi_x, \varphi_y \rangle_{\mathcal{H}^*} = \langle y, x \rangle_{\mathcal{H}}$$

Exercise 12.2 *Let $A \subset \mathcal{H}$ be such that*

$$\mathcal{H}_0 = \{x \in \mathcal{H} \mid x \textit{ is a finite linear combination of elements in } A\},$$

is a dense linear subspace of $\mathcal{H}$. Prove that any bounded linear functional on $\mathcal{H}$ is uniquely determined by its values on A. In other words, show that if $\varphi, \xi \in \mathcal{H}^*$ such that $\varphi(a) = \xi(a)$, for all $a \in A$, then $\varphi = \xi$.

Note that that any o.n. basis $\{u_\alpha\}$ of $\mathcal{H}$ has the property required of the set A in the previous exercise. Hence, bounded linear functionals are uniquely determined by their values on an o.n. basis. What is a necessary and sufficient condition on a family of complex numbers c_α for there to be a bounded linear functional φ such that $\varphi(u_\alpha) = c_\alpha$?

Exercise 12.3 *Let $\{u_\alpha\}$ be an o.n. basis of $\mathcal{H}$. prove that $\{\varphi_{u_\alpha}\}$ is an o.n. basis of $\mathcal{H}^*$.*

Example:
A *state* ω of a quantum mechanical system, in general, is a linear functional ω on a space $\mathcal{A}$ of linear operators on a Hilbert space $\mathcal{H}$ ($\mathcal{A}$ represents the *observables* of the system), that satisfies the following two properties:

1. $\omega(A^*A) \geq 0$, for all $A \in \mathcal{A}$ (*positivity*).

2. $\omega(\mathbb{1}) = 1$ (*normalization*).

For simplicity suppose $\mathcal{H} = \mathbb{C}^n$, and $\mathcal{A}$ is the space of all $n \times n$ matrices with complex entries, acting as linear transformations on $\mathcal{H}$. Recall that $\mathcal{A}$ is a Hilbert space with the inner product given by

$$\langle A, B \rangle = \mathrm{Tr} A^* B$$

By the Riesz Representation Theorem 12.2 for each ω there is a unique $\rho \in \mathcal{A}$ such that

$$\omega(A) = \mathrm{Tr}\rho^* A, \qquad \text{for all } A \in \mathcal{A}$$

The conditions 1) and 2) translate into $\rho \geq 0$, and $\mathrm{Tr}\rho = 1$, respectively.

12.2 Adjoint operation on $\mathcal{B}(\mathcal{H})$

The most important application of the Riesz Representation Theorem is probably the definition of the adjoint operation of $\mathcal{B}(\mathcal{H})$. Our aim is to show that for any $A \in \mathcal{B}(\mathcal{H})$, there is a unique $A^* \in \mathcal{B}(\mathcal{H})$, such that

$$\langle x, Ay\rangle = \langle A^*x, y\rangle \tag{12.6}$$

The uniqueness of A^*, if it exists, follows from the following exercise.

Exercise 12.4 a) *Suppose $A, B \in \mathcal{B}(\mathcal{H})$ satisfy*

$$\langle x, Ay\rangle = \langle x, By\rangle, \quad \textit{for all } x, y \in \mathcal{H} \quad .$$

Prove that it follows that $A = B$.

b) *Use the result of art a) and polarization to prove that if*

$$\langle x, Ax\rangle = \langle x, Bx\rangle, \textit{for all } x,$$

then, $A = B$.

To prove that A^* exists means to show that for every $x \in \mathcal{H}$, there is a vector $z \in \mathcal{H}$, depending linearly on x, and such that

$$\langle z, y\rangle = \langle x, Ay\rangle \tag{12.7}$$

Observe that, for x fixed, the map φ_x defined by

$$\varphi_x(y) = \langle x, Ay\rangle$$

is a linear functional on $\mathcal{H}$, and that $\|\varphi_x\|$ is bounded by $\|x\|\|A\|$. By the Riesz Representation Theorem there is then a unique z satisfying (12.7). The linearity follows from the uniqueness in the Riesz Representation Theorem and the linearity of the inner product.

Exercise 12.5 *Prove the following properties of the adjoint: for all $A, B \in \mathcal{B}(\mathcal{H})$, and $\lambda \in \mathbb{C}$ one has*

a) $A^{**} = A$.

b) $(AB)^* = B^*A^*$.

c) $(\lambda A)^* = \overline{\lambda}A^*$.

d) $(A + B)^* = A^* + B^*$.

e) $\|A^*\| = \|A\|$.

f) $\|A^*A\| = \|A\|^2$.

Exercise 12.6 *Show by an example that, in general, $\|AA\| \neq \|A\|^2$.*

12.3 Weak convergence in a Hilbert space

Let $\mathcal{H}$ be a Hilbert space and (x_n) a sequence in $\mathcal{H}$. We say that x_n converges *weakly* to $x \in \mathcal{H}$ iff

$$\lim_{n\to\infty} \langle y, x_n \rangle = \langle y, x \rangle, \qquad \text{for all } y \in \mathcal{H}$$

Weak convergence for sequences in a Hilbert space is a special case of weak convergence in a Banach space. Let X be a Banach space, and (x_n) a sequence in X. We say that x_n converges weakly to $x \in X$ iff $\lim_n \varphi(x_n) = \varphi(x)$ for all $\varphi \in X^*$. In the case that X is a Hilbert space, why is this equivalent to the definition given above for weak convergence in a Hilbert space?

It is easy to see that, in any Banach space, weak convergence is weaker than norm convergence in the sense that if $x_n \to x$ in norm, then $x_n \to x$ weakly. This follows from

$$|\varphi(x_n) - \varphi(x)| = |\varphi(x_n - x)| \le \|\varphi\| \|x_n - x\|$$

The converse is not true whenever the Hilbert space is infinite-dimensional. This follows from the next exercise.

Exercise 12.7 *Let $\{u_n\}$ be a sequence of orthonormal vectors in a Hilbert space. Prove that $u_n \to 0$ weakly.*

Exercise 12.8 *Prove that in a finite-dimensional Hilbert space (e.g., $\mathbb{C}^n$) norm and weak convergence are equivalent.*

Exercise 12.9 *Let $\mathcal{H}$ be a Hilbert space, $\{u_\alpha\}$ an o.n. basis of $\mathcal{H}$, and (x_n) a sequence in $\mathcal{H}$. Suppose there is $x \in \mathcal{H}$ such that*

$$\langle u_\alpha, x_n \rangle \to \langle u_\alpha, x \rangle, \qquad \textit{for all } \alpha$$

Prove or disprove that then necessarily $x_n \to x$ weakly.

Theorem 12.3 *Let (x_n) be a sequence in $\mathcal{H}$ converging weakly to x. If in addition*

$$\lim_{n\to\infty} \|x_n\| = \|x\|$$

then $x_n \to x$ in norm.

Proof: We start from

$$\|x_n - x\|^2 = \|x_n\|^2 + \|x\|^2 - 2\mathrm{Re}\langle x_n, x \rangle$$

If $x_n \to x$ weakly we have, in particular,

$$\lim_n \langle x_n, x \rangle = \langle x, x \rangle$$

and, hence, under the hypotheses of the theorem, $\lim_n \|x_n - x\|^2 = 0$. ∎