15 Examples of compact linear operators

Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$. Before one can apply the spectral theorem in the form we have derived, one has to check that A is compact. There are many ways of verifying that A is compact. The following is a partial list:

1. Check the definition (Definition 14.1 directly. See Section 15.1).
2. Show that A is Hilbert-Schmidt (see Section 15.2).
3. Show that A is the norm limit of a sequence of compact (e.g., finite-rank) operators (Proposition 14.2).
4. Show that there is a self-adjoint compact operator B, and a function $f : \mathbb{R} \to \mathbb{C}$, continuous at 0, such that $A = f(B)$ (see Section 16.1).

15.1 Compactness in Hilbert space

One way of proving that A is compact would be by direclty verifying the definition: prove that for any bounded set $B \subset \mathcal{H}$, the set $A(B) = \{Ax \mid x \in B\}$, is precompact. A characterization of precompact sets in a Hilbert space is given by the following theorem.

Theorem 15.1 *Let $\mathcal{H}$ be a Hilbert space, and let C be a subset of $\mathcal{H}$.*
i) If C is precompact one has that for any o.n. set $\{e_n\}$ the following property holds: for all $\epsilon > 0$, there exists N such that

$$\sum_{n \geq N} |\langle e_n, x\rangle|^2 < \epsilon, \qquad \textit{for all } x \in C \tag{15.1}$$

ii) If C is bounded and there is an o.n. basis $\{e_n\}$ such that for all $\epsilon > 0$, there exists N such that (15.1) holds, then C is pre-compact.

Proof: *Proof of i):* It is easy to show that if C is pre-compact, then it is necessarily bounded (as in any normed space). Suppose C is bounded and such that (15.1) does not hold. We will prove that C cannot be precompact.

If (15.1) does not hold, there is $\epsilon > 0$, such that for all N, there is $x_N \in C$ such that

$$\sum_{n > N} |\langle e_n, x_N\rangle|^2 \geq \epsilon \tag{15.2}$$

We construct a subsequence of (x_N) as follows: let $n_1 = 1$. Then, pick n_2 such that

$$\sum_{n > n_2} |\langle e_n, x_{n_1}\rangle|^2 \leq \epsilon/4$$

etc. Given n_k, pick n_{k+1} such that

$$\sum_{n>n_{k+1}} |\langle e_n, x_{n_k}\rangle|^2 \leq \epsilon/4 \tag{15.3}$$

For any $n \geq 1$ define the orthogonal projection

$$P_n = \sum_{j\leq n} \langle e_j, \cdot\rangle e_j$$

The sequence (x_{n_k}) does not have any Cauchy subsequence: for $k > l$ we have

$$\begin{aligned}
\|x_{n_k} - x_{n_l}\|^2 &\geq \|(\mathbb{1} - P_{n_k})(x_{n_k} - x_{n_l})\|^2 \\
&\geq [\|(\mathbb{1} - P_{n_k})x_{n_k}\| - \|(\mathbb{1} - P_{n_k})x_{n_l})\|]^2 \\
&= \left[\sqrt{\sum_{n>n_k} |\langle e_n, x_{n_k}\rangle|^2} - \sqrt{\sum_{n>n_k} |\langle e_n, x_{n_l}\rangle|^2}\right]^2 \\
&\geq (\epsilon^{1/2} - (\epsilon/4)^{1/2})^2 \geq \frac{\epsilon}{4}
\end{aligned}$$

where we have used (15.2), $n_k \geq n_{l+1}$, and (15.3). It follows that (x_{n_k}) does not have any convergent subsequence, contradicting the hypothesis that C is precompact.
Proof of ii): Let $\{e_n\}$ be an o.n. basis with the stated property, and let (x_n) be any sequence in C. We will use the diagonalization trick to construct a convergent subsequence, thus proving that C is precompact. Without loss of generality we may assume that $\|x\| \leq 1$, for all $x \in C$. Start with $n_1 = 1$. We have that

$$\|(\mathbb{1} - P_{n_1})x_n\| \leq 1, \quad \text{for all } n$$

As all $P_{n_1}x_n$ are in the finite-dimensional Hilbert subspace spanned by $e_1, \ldots, e_{n_1}$, there is subsequence $(x_{1,k})$ such that that $P_{n_1}x_{1,k}$ converges. Therefore, we can pick the subsequence such that (see Excercise 15.1)

$$\|P_{n_1}(x_{1,k} - x_{1,l})\|^2 \leq \frac{1}{k}$$

for all $l \geq k$. Next pick n_2 such that

$$\|(\mathbb{1} - P_{n_2})x_{1,k}\|^2 \leq \frac{1}{2}, \quad \text{for all } k$$

This is possible due to the assumption (15.1). Then pick a subsequence $x_{2,k}$ of $x_{1,k}$, such that $P_{n_2}x_{2,k}$ is Cauchy and such that

$$\|P_{n_2}(x_{2,k} - x_{2,l})\|^2 \leq \frac{1}{k}$$

for all $l \geq k$. Etc: take n_l such that

$$\|(\mathbb{1} - P_{n_l})x_{l-1,k}\|^2 \leq \frac{1}{l}, \quad \text{for all } k$$

and then pick a subsequence $x_{l,k}$ of $x_{l-1,k}$, such that $P_{n_l}x_{l,k}$ satisfies:

$$\|P_{n_l}(x_{l,k} - x_{l,r})\|^2 \leq \frac{1}{k}$$

for all $r \geq k$. The diagonal sequence $x_{k,k}$ is then Cauchy:

$$\|x_{m,m} - x_{n,n}\|^2 = \|P_k(x_{m,m} - x_{n,n})\|^2 + \|(\mathbb{1} - P_k)x_{m,m} - x_{n,n}\|^2 \leq \frac{2}{k}$$

for all $m, n \geq k$. ■

In the proof above we have used the following excercise. If it is not entirely obvious to you, you should write a proof of this statement:

Exercise 15.1 *Let (X, d) be a metric space, and let (x_n) be a sequence in X. Prove that if (x_n) has a Cauchy subsequence, then, for any decreasing sequence of positive $\epsilon_k \to 0$, (x_n) has a subsequence (x_{n_k}) such that*

$$d(x_{n_k}, x_{n_l}) \leq \epsilon_k, \quad \textit{for all } l > k$$

15.2 Hilbert-Schmidt operators

Any finite-rank operator ($\dim \mathrm{ran} A < \infty$), is compact. From Proposition 14.2 we know that uniform limits of compact operators are compact. Therefore we can prove that A is compact by showing that there is a sequence of finite-rank operators A_n that converges to A in norm.

E.g., this way we can prove that any operator on $L^2(D)$, $D \subset \mathbb{R}^n$, of the form

$$Kf(x) = \int_D k(x, y) f(y)\, dy, \quad \text{for all } f \in L^2(D) \tag{15.4}$$

with $k \in \mathcal{L}^2(D \times D)$, is compact. These operators are called *Hilbert-Schmidt* operators.

Exercise 15.2 *Consider the operator K defined in (15.4).*

a) *Prove that K is compact if $k \in \mathcal{L}^2(D \times D)$.*

b) *Show that K^* is also given as an integral operator of the form*

$$Kf(x) = \int_D l(x, y) f(y)\, dy$$

and find its kernel l in terms of k.

More generally, suppose $\mathcal{H}$ is a separable Hilbert space, and $A \in \mathcal{B}(\mathcal{H})$. If there is an o.n. basis $\{e_n\}$ such that

$$\sum_n |Ae_n\|^2 < +\infty$$

A is said to be Hilbert-Schmidt, and the quantity

$$\|A\|_{HS} = \sqrt{\sum_n \|Ae_n\|^2} \tag{15.5}$$

is the *Hilbert-Schmidt norm* of A.

Exercise 15.3 *Let $\mathcal{H}$ be a separable Hilbert space with an o.n. basis $\{e_n\}$, and $A \in \mathcal{B}(\mathcal{H})$ such that*

$$\sum_n \|Ae_n\|^2 < +\infty$$

a) *Prove that the Hilbert-Schmidt norm defined in (15.5) is independent of the basis. I.e., show that for any other o.n. basis $\{f_n\}$ one has*

$$\sum_n \|Af_n\|^2 = \sum_n \|Ae_n\|^2$$

b) *Prove that*

$$\|A\|_{HS} = \|A^*\|_{HS}$$

15.3 Sturm-Liouville problems

When we studied Fourier series we effectively diagonalized the unbounded operator Δ (called the *Laplacian*, or *Laplace operator*) given by

$$\Delta f = \frac{d^2}{dx^2} f$$

Indeed the o.n. basis of $L^2([0, 2\pi])$, defined by $e_n(x) = e^{inx}$, $n \in \mathbb{Z}$, is a basis of eigenvectors of Δ:

$$\Delta e^{inx} = -n^2 e^{inx}$$

So, the eigenvalues are $-n^2$ (which are unbounded, showing that Δ is not a bounded linear operator). The inverse operator of Δ is not well-defined as the constant functions are in the kernel of Δ. The inverse of $\Delta - \lambda$, with $\lambda \neq -n^2$, for any $n \in \mathbb{Z}$, is defined by

$$(\Delta - \lambda)^{-1} = -\sum_{n \in \mathbb{Z}} (n^2 + \lambda)^{-1} \langle e_n, \cdot \rangle e_n$$

This is a special case of a more general situation.

Let p, q, and w be real-valued functions defined on an interval $[a, b]$ and satisfying: $q, w \in C([a, b])$, $p \in C^1([a, b])$, $w(x) > 0, p(x) > 0$, for all $x \in [a, b]$.

Let H be the Hilbert space of functions f satisfying

$$\int_a^b |f(x)|w(x)dx < \infty$$

with inner product

$$\langle f, g \rangle = \int_a^b \overline{f(x)} g(x)w(x)dx$$

The differential operator given by

$$Lf = \frac{1}{w}[-(pf')' + qf)] \tag{15.6}$$

for f such that $Lf \in H$ is well-defined is called a *Sturm-Liouville* operator. Next quarter, we will be more precise about the definition of unbounded operators such as L. In particular, we will then explain why (15.6) should be supplemented by boundary conditions such as

$$\begin{aligned} Af(a) + A'f'(a) &= 0 & (15.7) \\ Bf(b) + B'f'(b) &= 0 & (15.8) \end{aligned}$$

where A, A', B, B' are real constants such that A and A' are not both 0, and B and B' are not both 0.

Theorem 15.2 *Let L be defined as in (15.6) with the boundary conditions (15.8) and assume 0 is not an eigenvalue of L. Then there exists a continuous function $g : [a, b] \times [a, b] \to \mathbb{R}$, satisfying $g(x, y) = g(y, x)$, such that*

$$L^{-1}f(x) = \int_a^b g(x, y)f(y)dy \tag{15.9}$$

defines the inverse operator.

For a proof of this theorem and more details on Sturm-Liouville operators, see MAT203C next quarter. Note that Proposition 5.2 of MAT203A is a special case of the above theorem.

It follows from (15.9) that L^{-1} is Hilbert-Schmidt and hence compact. And as g is real and symmetric, it is self-adjoint. Therefore, there exists an o.n. basis of eigenvectors $\{e_n\}$ of L^{-1}. $\{e_n\}$ is then also an o.n. basis of eigenvectors of L (with the inverse eigenvalues).

Note that if 0 is an eigenvalue of L, one can apply the theorem to the operator $L - \lambda \mathbb{1}$, with $\lambda \in \mathbb{R}$ chosen such that 0 is not an eigenvalue of $L - \lambda \mathbb{1}$. The relation between L and $L - \lambda \mathbb{1}$ is a redefinition of the function q:

$$(L - \lambda) f = \frac{1}{w} \left[-(pf')' + (q - \lambda w) f) \right]$$