18 Distributions

Our next two topics are *distributions* and the *Fourier transform*. Both will allow us to study, compute, and use Green's functions more easily. In addition, both topics have many other applications.

18.1 The Schwartz space $\mathcal{S}$

Recall the definition of *seminorm*: a function $p : X \to \mathbb{R}$, where X is a linear space over $\mathbb{C}$, is a seminorm on X if it has the following three properties:

- $p(x) \geq 0$, for all $x \in X$.
- $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$.
- $p\lambda x) = |\lambda| p(x)$, for all $x \in X$, $\lambda \in \mathbb{C}$.

Let n be a positive integer. We will use the following notations: for $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$, $\alpha_i, \beta_i \geq 0$, for $i = 1, \ldots, n$

$$\partial^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \tag{18.1}$$

$$x^\alpha := \prod_{i=1}^n x_i^{\alpha_i} \tag{18.2}$$

$$\alpha + \beta := (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n) \tag{18.3}$$

$$|\alpha| := \sum_{i=1}^n |\alpha_i| = \sum_{i=1}^n \alpha_i \tag{18.4}$$

$$\alpha! := \prod_{i=1}^n \alpha_i! \tag{18.5}$$

$$\alpha \geq \beta \quad \text{iff} \quad \alpha_i \geq \beta_i, i = 1, \ldots, n \tag{18.6}$$

With these notations Taylor's remainder theorem and Leibniz' rule can be written in the following compact form: for any $f \in C^k(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$

$$f(x) = \sum_{0 \leq |\alpha| \leq k} \partial^\alpha f(x_0) \frac{(x - x_0)^\alpha}{\alpha!} + R_k(x) \tag{18.7}$$

$$\text{with } \lim_{x \to x_0} \frac{R_k(x)}{\|x - x_0\|^k} = 0 \tag{18.8}$$

$$\partial^\alpha (fg) = \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta! \gamma!} \left(\partial^\beta f\right) \left(\partial^\gamma g\right) \tag{18.9}$$

The seminorms we will use in this section are defined on C^∞ functions f. For all $n \geq 0, \alpha \in (\mathbb{Z}_+)^n$, define

$$p_{n,\alpha}(f) = \sup_{x \in \mathbb{R}^n} (1 + \|x\|)^n \, |\partial^\alpha f(x)| \tag{18.10}$$

The space of C^∞ functions for which all the seminorms $p_{n,\alpha}$ are finite is called *Schwartz space*, and is conventionally denoted by $\mathcal{S}(\mathbb{R}^n)$. We will often simply write $\mathcal{S}$ instead of $\mathcal{S}(\mathbb{R}^n)$, and $\|f\|_{n,\alpha}$ instead of $p_{n,\alpha}$. The elements of $\mathcal{S}$ are often called *test functions*, which is less specific than *Schwartz functions.*

Exercise 18.1 *Prove that the seminorms $p_{n,\alpha}$ are in fact norms on $\mathcal{S}$.*

There are plenty of functions in $\mathcal{S}$. E.g., every functions of the form $Q(x)\exp(-c\|x-x_0\|^2)$, with Q a polynomial, $c > 0$, and $x_0 \in \mathbb{R}$, is a Schwartz function.

Exercise 18.2 *Check that $\mathcal{S}$ is an algebra for the usual operations of addition and multiplication of functions.*

In general, if $\{p_a\}_{a\in A}$ is family of seminorms on a linear space X, indexed by A, X becomes a topological vector space with the following definition of a base of open neighborhoods $\mathcal{N}$:

$$\begin{aligned} \mathcal{N} &= \{N_{x,\{a_i\},\epsilon} \mid x \in X, \{a_i\} \subset A \text{ finite}, \epsilon > 0\} \\ N_{x,\{a_i\},\epsilon} &= \{y \in X \mid p_a(x-y) < \epsilon, \text{ for all } a \in \{a_i\}\} \end{aligned}$$

The following three exercises are general properties topological vector spaces defined using a family of seminorms that we will need for our study of distributions.

Exercise 18.3 *Prove the following:*

a) *The addition of vectors in X and the multiplication by a scalar are continuous.*

b) *A topology defined by a family of seminorms as above has a base of convex open neighborhoods. Such a topological space is called* locally convex*.*

c) *If for all $x \in X$ there exists $a \in A$ such that $p_a(x) > 0$, then the topology defined by $\{p_a\}_{a\in A}$ is Haussdorff.*

If the family of seminorms is countable, the topology defined by it is *metric*, meaning that there is a metric d on X that defines the same open sets. One also says that X (or its topology) is *metrizable.*

Exercise 18.4 *Suppose that A is countable, and let $p_1, p_2, p_3, \ldots$ be an enumeration of the $\{p_a\}$.*

a) *Prove that*

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)} \tag{18.11}$$

defines a metric on X.

b) *Prove that the topology defined by* d *coincides with the one defined by* $\{p_a\}_{a\in A}$.

Note that the set of seminorms $\{p_{n,\alpha}\}$ on $\mathcal{S}$ is countable. Hence, by the previous exercise, $\mathcal{S}$ is a metric space. The convergence of a sequence (x_n) to a limit $x \in X$ can be expressed in terms of the metric, but it is often more convenient to express it in terms of the seminorms, i.e., $x_n \to x$ iff for all $a \in A$, $p_a(x_n - x) \to 0$.

Exercise 18.5 *Let* (x_n) *be a sequence in* X. *Prove that* (x_n) *is a Cauchy sequence for the metric* d *defined in (18.11) iff for all* $a \in A$, *and* $\epsilon > 0$ *there exists* N *such that* $p_a(x_n - x_m) < \epsilon$ *for all* $n, m \geq N$.

In the following proposition we prove that $\mathcal{S}$ is a *complete* metric space.

Proposition 18.1 $\mathcal{S}$ *with the topology defined above is a complete metric space.*

Proof: Let (f_n) be a Cauchy sequence in $\mathcal{S}$. We have to prove that (f_n) converges for the topology on $\mathcal{S}$ to a function $f \in \mathcal{S}$. We start from the fact that the bounded continuous functions with the sup norm are complete. From the previous exercise we know that (f_n) Cauchy in the topology of $\mathcal{S}$ implies that it is a Cauchy sequence for the sup norm ($p_{0,0}$ is the sup norm). Therefore, there exists a bounded continuous function f such that $f_n \to f$ in the sup norm. Clearly, f is the candidate limit for f_n in $\mathcal{S}$. Two questions remain: is f in $\mathcal{S}$, and does (f_n) converge to f in the topology of $\mathcal{S}$?

Again by the previous exercise we have that for all α the sequence $\partial^\alpha f_n$ is also Cauchy for the sup norm, and hence converges uniformly to a function g_α. We claim that

$$g_\alpha = \partial^\alpha f, \qquad \text{for all } \alpha. \tag{18.12}$$

Then, by using the fact that f_n is Cauchy for all the norms $p_{n,\alpha}$, we see that $f \in \mathcal{S}$ and $f_n \to f$ in $\mathcal{S}$.

We prove (18.12) for all α, by induction on $|\alpha|$. It holds for $|\alpha| = 0$. Suppose we have proved (18.12) for all α with $|\alpha| \leq m$. Then, if $|\beta| = m+1$, there exist α, $|\alpha| = m$, and j such that $\beta = \alpha + e_j$, where e_j is the jth standard basis vector of $\mathbb{Z}^n$, and we have by the Fundamental Theorem of Calculus that

$$\partial^\alpha f_n(x + te_j) - \partial^\alpha f_n(x) = \int_0^t \partial^{e_j} \partial^\alpha f_n(x + se_j)\, ds$$

Clearly, $\partial^{e_j}\partial^\alpha = \partial^\beta$. By taking $n \to \infty$ we obtain

$$\partial^\alpha f(x+te_j) - \partial^\alpha f(x) = \int_0^t g_\beta(x+se_j)\,ds$$

Next, take $\lim_{t\downarrow 0}$ of $(1/t)$ times this expression and using the definition of derivative and the continuity of g_β you find

$$\partial^\beta f(x) = g_\beta(x)$$

■

Summarizing, we have found that $\mathcal{S}$ with its topology defined by the seminorms $p_{n,\alpha}$, is a locally convex metrizable complete Haussdorff space. A topological linear space with these properties is called a *Fréchet space.*

Exercise 18.6 *Let $f \in C^\infty(\mathbb{R}^n)$. Then, $f \in \mathcal{S}$ iff $x^\alpha \partial^\beta f$ is bounded for all α, β iff $\partial^\beta(x^\alpha f)$ is bounded for all α, β.*

Exercise 18.7 *Let $k \in \mathcal{S}$ and define*

$$Kf(x) = \int k(x-y)f(y)\,dy, \quad \textit{for all } f \in \mathcal{S}$$

Prove that $K : \mathcal{S} \to \mathcal{S}$, which is called a convolution operator, is a continuous linear operator for the topology of $\mathcal{S}$.

Exercise 18.8 *For every $a \in \mathbb{R}^n$ define a linear transformation $T_a : \mathcal{S} \to \mathcal{S}$ by $T_a(f)(x) = f(x-a)$. These are the translation operators on $\mathcal{S}$.*

a) *Prove that for all $a \in \mathbb{R}^d$, T_a is continuous for the topology of $\mathcal{S}$.*

b) *Prove that for all $f \in \mathcal{S}$, the map $a \mapsto T_a f$ is continuous from $\mathbb{R}^d$ to $\mathcal{S}$. (Hint: First, prove that for $f \in C(\mathbb{R}^n)$ one has $\lim_{a\to 0} \|T_a f - f\|_{\sup} = 0$ iff f is uniformly continuous. Also, note that it is sufficient to prove continuity in $a = 0$, due to the group property of T_a.)*

18.2 The space $\mathcal{S}^*$, the tempered distributions

The space of all continuous linear functionals on $\mathcal{S}$ is denoted by $\mathcal{S}^*$ (or $\mathcal{S}'$). It is the topological dual of $\mathcal{S}$. The elements of $\mathcal{S}^*$ are called *tempered distributions.*

As $\mathcal{S}$ is a metric space, we can prove that a functional $\varphi : \mathcal{S} \to \mathbb{C}$ is continuous by verifying that for every convergent sequence $f_n \to f$ in $\mathcal{S}$ we have

$$\lim_n \varphi(f_n) = \varphi(f)$$

Example:
The prime example is the so-called delta-function (which is a bit of a misnomer because it is not a function on $\mathbb{R}^n$, but a functional on $\mathcal{S}$). For every $x_0 \in \mathbb{R}^n$ define a linear functional δ_{x_0} on $\mathcal{S}$ by

$$\delta_{x_0}(f) = f(x_0), \quad f \in \mathcal{S}$$

If $f_n \to f$ in $\mathcal{S}$, then, in particular we know that $f_n \to f$ uniformly. Therefore, $f_n(x_0) \to f(x_0)$, which proves that δ_{x_0} is continuous. The linearity of δ_{x_0} is trivial. We conclude that $\delta_{x_0} \in \mathcal{S}^*$.

The action of the elements of the dual X^* on X is often represented by a *pairing*, which can be viewed as a generalization of an inner product.

$$\langle \cdot, \cdot \rangle : \mathcal{S}^* \times \mathcal{S} : \langle \varphi, f \rangle = \varphi(f)$$

Note that, due to the Riesz Representation Theorem, in the case where $X = \mathcal{H}$ is a Hilbert space the pairing on $\mathcal{H}^* \times \mathcal{H}$ can be indentified with the inner product on $\mathcal{H}$.

A popular notation for the pairing on $\mathcal{S}^* \times \mathcal{S}$ is given by

$$\varphi(f) = \langle \varphi, f \rangle = \int \varphi(x) f(x)\, dx$$

as if $\mathcal{S}^*$ were a function space. This notation is justified by the fact that the pairing on $\mathcal{S}^* \times \mathcal{S}$ shares a number of properties with inner products defined through an integral. For the same reason it often is a convenient notation in computations, but keep in mind that it is just a notation for continuous linear functionals. The distribution δ_{x_0} is then written as

$$\delta_{x_0}(f) = \int \delta(x - x_0) f(x)\, dx = f(x_0)$$

See the representation theorem 18.2 later for further justification of this notation.
More examples:

1. Let F be a continous function on $\mathbb{R}^n$ of at most polynomial growth, i.e., there exists p and $C > 0$ such that

$$|F(x)| \leq C(1 + \|x\|)^p$$

   Define $\varphi_F \in \mathcal{S}^*$ by

$$\varphi_F(f) = \int_{\mathbb{R}^n} F(x) f(x)\, dx$$

2. If $K : \mathcal{S} \to \mathcal{S}$ is a continuous linear transformation of $\mathcal{S}$, and $\varphi \in \mathcal{S}^*$, then $\varphi \circ K \in \mathcal{S}^*$.

3. An important case of the previous example is $K = \partial^\alpha$. So, e.g.,

$$f \mapsto \delta_{x_0}(\partial^\alpha f) = \partial^\alpha f(x_0)$$

is a functional in $\mathcal{S}^*$. The following is a standard notation inspired by the Integration by Parts formula for true integrals:

$$\begin{aligned} \delta_{x_0}(\partial^\alpha f) &= \int \delta(x - x_0)\partial^\alpha f(x)\, dx \\ &= (-1)^{|\alpha|} \int (\partial^\alpha \delta)(x - x_0) f(x)\, dx \\ &= (-1)^{|\alpha|} \int \delta^{(\alpha)}(x - x_0) f(x)\, dx \end{aligned}$$

So, in one dimension ($n = 1$), δ'_{x_0} denotes the functional on $\mathcal{S}(\mathbb{R})$ defined by

$$\delta'_{x_0}(f) = -f'(x_0)$$

and similarly

$$\delta''_{x_0}(f) = f''(x_0)$$

This notation can be generalized to arbitrary tempered distributions: one defined the parial derivative $\partial^\alpha \varphi$, for any $\varphi \in \mathcal{S}^*$ by

$$\partial^\alpha \varphi(f) = (-1)^{|\alpha|} \varphi(\partial^\alpha f)$$

4. For any $g \in L^1(\mathbb{R}^d)$, φ_g defined by

$$\varphi_g(f) = \int g(x) f(x)\, dx, \quad f \in \mathcal{S}$$

is a tempered distribution. Moreover, φ_g is uniquely determined by $g \in L^1$. This can be seen as follows. Suppose $g_1, g_2 \in L^1(\mathbb{R}^d)$. Then $h = g_1 - g_2 \in L^1(\mathbb{R}^d)$. We have to show that if $\varphi_{g_1} = \varphi_{g_2}$, then $h = 0$. The latter can be proved by showing that

$$\int_A h(x)\, dx = 0 \tag{18.13}$$

for all sets $A \subset \mathbb{R}^d$ of the form $[a_1, b_1] \times \cdots [a_d, b_d]$. (18.13) follows from $\varphi_{g_1} = \varphi_{g_2}$ by approximating the characteristic function χ_A with a sequence of test functions f_n:

$$\int_A h(x)\, dx = \lim_{n\to\infty} \int h(x) f_n(x) = \lim_{n\to\infty} \varphi_{g_1}(f_n) - \varphi_{g_2}(f_n) = 0$$

Exercise 18.9 *Let $\theta : \mathbb{R} \to \mathbb{R}$ denote the Heaviside function defined by*

$$\theta(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

For any $x_0 \in \mathbb{R}$, let Θ be the tempered distribution defined by

$$\Theta_{x_0}(f) = \int \theta(x - x_0) f(x0\, dx$$

Prove that $\Theta'_{x_0} = \delta_{x_0}$.

18.3 Convergence of tempered distributions

Let (φ_n) be a sequence in $\mathcal{S}^*$. We will say that (φ_n) converges to $\varphi \in \mathcal{S}^*$ iff

$$\lim_{n\to\infty} \varphi_n(f), \qquad \text{for all } f \in \mathcal{S} \tag{18.14}$$

Note that for each $f \in \mathcal{S}$ the map

$$\varphi \mapsto \varphi(f) \tag{18.15}$$

is a linear functional on $\mathcal{S}$. The notion of convergence of distributions we just introduced corresponds to a natural topology on the tempered distributions defined as the weakest topology for which all the functionals of the form (18.15) are continuous. This topology is called the *weak-* topology*, often abreviated as w^*-topology. It can also be viewed as a topology defined by the family of seminorms $\{p_f \mid f \in \mathcal{S}\}$, where ,

$$p_f(\varphi) = |\varphi(f)|$$

Exercise 18.10 *Prove that p_f is a seminorm on $\mathcal{S}^*$.*

A particularly important kind of sequences of distributions are the ones that converge to δ_{x_0}. They are called *delta sequences.* One is usually interested in approximating δ_{x_0} by distributions of the form

$$\varphi_n(f) = \int \delta_n(x - x_0) f(x)\, dx \tag{18.16}$$

where δ_n is a true function with nice properties, e.g., $\delta_n \in \mathcal{S}$.
Examples of delta sequences:

1. By translation a sequence converging to δ_{x_0} can be obtained from a sequence converging to δ_0. Therefore we will just give examples for delta sequences converging to δ_0. You already encountered several delta sequences without thinking of them in terms of distributions. E.g., in one dimension,

$$\varphi_n(f) = \frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} f(x)\,dx \to \delta_0(f) = f(0)$$

which is of the form (18.16) with

$$\delta_n(x) = \begin{cases} n/2 & \text{if } -1/n \le x \le 1/n \\ 0 & \text{elsewhere} \end{cases}$$

2. The approximation by Gaussians is another example we are already familiar with:

$$\delta_n(x) = \sqrt{\frac{n}{\pi}}e^{-nx^2} \tag{18.17}$$

3. The following is also a useful delta sequence:

$$\delta_n(x) = \frac{\sin nx}{\pi x} \tag{18.18}$$

Exercise 18.11 a) *Prove that*

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x}\,dx = \pi$$

(Hint: use Cauchy's formula applied to the function $\exp(iz)$*.)*
b) *Prove that the δ_n of (18.18) indeed defines a delta sequence.*

It is obvious that $\mathcal{S}$ can be continuously embedded in $\mathcal{S}^*$, and it is also clear that this embedding will not be onto. Therefore, we cannot expect to have a strict analogue of the Riesz Representation theorem for the continuous linear functionals on a Hilbert space. But there is an analogous theorem that we state here without proof.

Theorem 18.2 *For every $\varphi \in \mathcal{S}^*$ there is a continuous function $F : \mathbb{R}^n :\to \mathbb{C}$ of at most polynomial growth and a multi-index α such that for all $f \in \mathcal{S}$*

$$\varphi(f) = \int F(x)\partial^\alpha f(x)\,dx$$

18.4 Convolution

By Exercise 18.7 we know that for $f, g \in \mathcal{S}$, the function $f * g$ defined by

$$x \mapsto f * g(x) := \int_{\mathbb{R}^d} f(x-y)g(y)\,dy \tag{18.19}$$

is in $\mathcal{S}$ and depends continuously on g (and f). $f * g$ is called the *convolution* of f and g. Using the translation operator defined in Exercise 18.8, and the reflection operator R defined by

$$(Rf)(x) = f(-x)$$

this can also be written as

$$f * g(x) := \int_{\mathbb{R}^d} (RT_x f)(y)g(y)\,dy$$

The following properties are easy to prove.

Proposition 18.3 *For any $f, g, h \in \mathcal{S}$ one has*
i) $f * g = g * f$.
ii) $(f * g) * h = f * (g * h)$.
iii) $T_a(f * g) = (T_a f) * g = f * (T_a g)$, *for all* $a \in \mathbb{R}^d$.

It is clear from (18.19) that the definition of convolution can be extended from test-functions to more general functions as long the integrals exist. E.g. the convolution of two L^1 functions is well-defined and is also L^1. It is also easy to see that if $f \in L^1$ and $g \in C^k$ with all derivatives up to order k bounded, then $f * g \in C^k$.

One can even define the convolution of $\varphi \in \mathcal{S}^*$ and $f \in \mathcal{S}$ by

$$\varphi * f(x) = \varphi(RT_x f)$$

One can prove that $\varphi * f \in C^\infty(\mathbb{R}^d)$, and that $\varphi * f$ is of at most polynomial growth.

We state without proof the following inequality for the L^p norm of convolutions. In particular, this inequality shows that convolution is a continuous operation when defined on an appropriate choice of Banach spaces.

Proposition 18.4 (Young's inequality) *Suppose $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$, and $1 \leq, p, q, r \leq \infty$ such that $p^{-1} + q^{-1} = 1 + r^{-1}$. Then, $f * g \in L^r(\mathbb{R}^d)$, and*

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\,\|g\|_{L^q}$$