2 Continuous functions

For further reading and more exercises see
J.E. Marsden, *Elementary Classical Analysis*, 2nd ed., W.H. Freeman and Company, New York, 1993.
or
W. Rudin, *Principles of Mathematical Analysis*, 2nd ed. McGraw-Hill, New York, 1964

2.1 Continuity and semi-continuity

Let D be a subset of $\mathbb{R}$. A function $f : D \to \mathbb{R}$ is *continuous* at a point $x_0 \in D$ iff for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in D$, $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. Continuity at x_0 is thus the property that the value of a function f in a point x close to x_0, is close to the value of f in x_0. If f is not continuous at x one says that it is *discontinuous* at x.

It is now obvious how to define continuity of a function from one metric space into another. Let (X, d_X) and (Y, d_Y) be two metric spaces.

Definition 2.1 (continuity) *A function $f : X \to Y$ is continuous at $x_0 \in X$ iff for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$, $d_X(x, x_0) < \delta$ implies $d_Y(f(x), f(x_0)) < \epsilon$.*

Remark: In the sequel we will usually denote all metrics by d when it is clear from the context (e.g., the arguments of d) which metric is meant.

We say that f is continuous on $D \subset X$ if f is continuous at all points $x \in D$. Note that this is not the same as saying that f restricted to D is continuous, e.g., if D contains only one point any function f restricted to D is continuous.

There are always continuous functions on X. E.g., any constant function is continuous. A less trivial example is the following. For any $x_0 \in X$, the function f defined by $f(x) = d(x_0, x)$ is continuous on X.

Definition 2.2 (uniform continuity) *A function $f : X \to Y$ is uniformly continuous on X iff for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$, $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$.*

Note the subtle difference between this and the previous definition.

Examples:

1. A function $f : \mathbb{R}^n \to \mathbb{R}^m$, satisfying

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y), \qquad \text{for all } x, y \in \mathbb{R}^n, t \in [0,1]$$

   is called *affine*. Affine functions are uniformly continuous.

2. The function $x \mapsto x^2, x \in \mathbb{R}$, is continuous but not uniformly continuous.

A function $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}$ is an open set, is called *differentiable* at $x_0 \in D$, iff the following limit exists (and is finite):

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Exercise 2.1 *Prove that if f is differentiable at x_0, then f is continuous at x_0.*

A useful characterization of continuous functions, which could also serve as an equivalent definition, is the following.

Proposition 2.3 (say it with sequences) *$f : X \to Y$ is continuous at $x \in X$ iff for every convergent sequence $x_n \to x$, the sequence $f(x_n)$ converges to $f(x)$.*

Proof: We will denote the metrics of both X and Y by d. The arguments of d unambiguously indicate which metric is meant.
The 'if' part: we show by construction that, if f is discontinuous at x, there is sequence (x_n), converging to x, but such that the sequence $(f(x_n))$ does not converge to $f(x)$. If f is discontinuous at x, there exists $\epsilon > 0$ such that for all $n \geq 1$, we can find $x_n \in X$ satisfying

$$d(x, x_n) < \frac{1}{n}, \text{ but } d(f(x), f(x_n)) > \epsilon$$

Clearly, the sequence x_n converges to x, but $f(x_n)$ does not converge to $f(x)$, as it stays at least a distance ϵ away from $f(x)$.
The 'only if' part: Suppose f is continuous at x, and $x_n \to x$. For any $\epsilon > 0$, there is $\delta > 0$ such that

$$\text{if } d(x, x_n) < \delta \text{ then } d(f(x), f(x_n)) < \epsilon$$

As $x_n \to x$, we can find N such that for all $n \geq N$, $d(x, x_n) < \delta$. Therefore, $f(x_n) \to f(x)$. ∎

From the fact that continuous functions preserve convergence of sequences it immediately follows that they also preserve compactness. This is expressed in the following theorem.

Theorem 2.4 *Let $f : K \to Y$ be continuous on K, where K is a compact metric space and Y is any metric space. Then, $f(K) \subset Y$ is compact.*

In the rest of this section the target space Y will always be taken to be the real numbers. A function $f : D \to \mathbb{R}$ is called *bounded* on D iff there exist $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in D$, which is equivalent to saying that $f(D)$ is a bounded set.

Here is our first practical use of compactness:

Theorem 2.5 *Let K be a compact metric space (e.g., a compact subset of a metric space). If $f : K \to \mathbb{R}$ is continuous, then f is bounded and attains its supremum (i.e, maximum) and infimum (i.e., minimum). I.e.,*

$$\sup_{x\in K} f(x) = f(y) \quad \textit{for some } y \in K \tag{2.1}$$

$$\inf_{x\in K} f(x) = f(z) \quad \textit{for some } z \in K \tag{2.2}$$

Proof: First, we show that f is bounded. By Theorem 2.4, $f(K)$ is a compact subset of $\mathbb{R}$. Therefore, $f(K)$ is bounded by the Heine-Borel Theorem 1.10.

As (2.1) follows from (2.2) applied to the function $-f$, we only need to prove (2.2). We already know that f is bounded. In particular, f is bounded from below, i.e., there exists a constant $M \in \mathbb{R}$ such that $f(x) \geq M$, for all $x \in K$. Hence, $m := \inf_{x\in K} f(x) \geq M > -\infty$. By the definition of infimum, for all n there is $x_n \in K$ such that

$$|m - f(x_n)| < \frac{1}{n}, \text{ and hence } f(x_n) \to m \tag{2.3}$$

As K is compact it follows that (x_n) has a convergent subsequence. Let (y_n) be such a subsequence and denote its limit by y. Clearly, (2.3) implies $f(y_n) \to m$. As f is continuous $f(y_n) \to f(y)$. By the uniqueness of the limit, Proposition 1.5, we conclude $f(y) = m$, i.e., f attains its infimum in the point y. ■

Examples:

1. The function $f(x) = 1 - \sqrt{|x|}$ is continuous on $[-1, 1]$ and bounded. f attains its maximum in $x = 0$. (The theorem applies)

2. $f(x) = e^x$ is continuous on the whole of $\mathbb{R}$ but not bounded. ($\mathbb{R}$ is not compact)

3. $f(x) = 1/x$ on $(0, 1]$ is continuous but not bounded. ($(0, 1]$ is not compact)

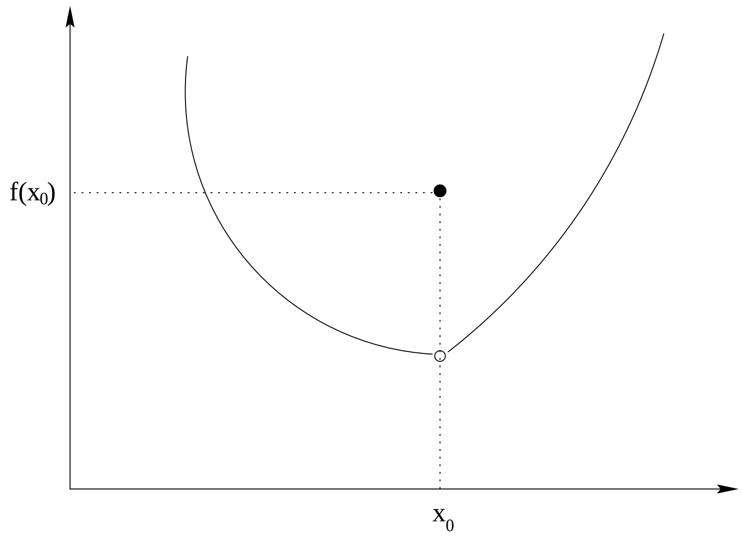

Figure 2.1: The function f is upper semicontinuous, but not continuous, in the point x_0.

4. $f(x) = \text{sign}x$ on $[-1, 1]$ is bounded, not continuous, but attains its maximum on any $x \in (0, 1]$ (by convention we put $\text{sign}0 = 0$). (continuity is not a necessary condition for attaining the maximum)

For functions that have only a certain kind of discontinuities, part of the conclusions of Theorem 2.5 still holds. There are two kinds of "half-continuous" functions, defined as follows.

Definition 2.6 (upper and lower semi-continuity) $f : X \to \mathbb{R}$ *is* upper *semi-continuous on X if for all $x \in X$ and any sequence $x_n \to x$, $x_n \neq x$, one has*

$$\limsup_{n\to\infty} f(x_n) \leq f(x)$$

Similarly, f is lower *semi-continuous on X if for all $x \in X$ and any sequence $x_n \to x$, $x_n \neq x$, one has*

$$\liminf_{n\to\infty} f(x_n) \geq f(x)$$

Note that f is continuous iff f is both upper and lower semicontinuous. The analogue of Theorem 2.5 is then:

Theorem 2.7 *Let K be a compact metric space. If $f : K \to \mathbb{R}$ is upper semi-continuous, then f is bounded from above and attains its supremum. If $f : K \to \mathbb{R}$ is lower semicontinuous, then f is bounded from below and attains its infimum.*

Bounded (from) above here means that there is $M \in \mathbb{R}$ such that $f(x) \le M$ for all $x \in K$, and f is *bounded (from) below* iff $-f$ is bounded from above.

Theorem 2.5 can be generalized in yet another way: by adding an additional assumption, called *coersiveness* , expressed by (2.4), we can also treat continuous functions on non-compact domains. See the following exercise.

Exercise 2.2 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying*

$$\lim_{\|x\| \to \infty} f(x) = +\infty \tag{2.4}$$

a) *Prove that f is bounded below.*

b) *Prove that f attains its infimum.*

Exercise 2.3 *Let $p : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function of two real variables. Suppose $p(x, y) \ge 0$ for all $x, y \in \mathbb{R}$. Do all such functions p attain their infimum? Prove or disprove.*

Exercise 2.4 *Let $(X, d_X), (Y, d_Y)$, and (Z, d_Z) be metric spaces and let $f : X \to Y$, and $g : Y \to Z$ be two continuous functions. Show that $h : X \to Z$, defined by $h(x) = g(f(x))$ is also continuous.*

2.2 Convergence of sequences of functions

In this section we consider sequences of real-valued functions on a common domain $D \subset X$ in a metric space (X, d), i.e., $f_n : D \to \mathbb{R}$, for $n \ge 1$. What would we mean by $f_n \to f$? Your "first thought" on this question is likely to be one of the following two:

1) The functions f_n are defined by the values they take on D. So, let us say that the functions converge if the values converge, i.e., if $f_n(x) \to f(x)$ for all $x \in D$. Thus, convergence of real-valued functions is reduced to convergence of real numbers, which we already know. This is called *pointwise* convergence of functions.

2) Find a set of functions with a reasonable metric on it and define $f_n \to f$ in the sense of that metric.

1) and 2) are both natural and useful ideas. But they are not compatible! It turns out that for most relevant domains D (e.g., whenever D is not a countable set), pointwise convergence cannot be expressed as convergence in a metric space (see later). This could be good motivation to study more general topological spaces (beyond metric spaces). Let us postpone this study for another while, and first consider 1) and 2), separately, in some more detail.

Besides being non-metric, pointwise convergence has another limitation: it does not preserve continuity. Consider, e.g., the functions $f_n : [0, 1] \to \mathbb{R}, f_n(x) = x^n$.

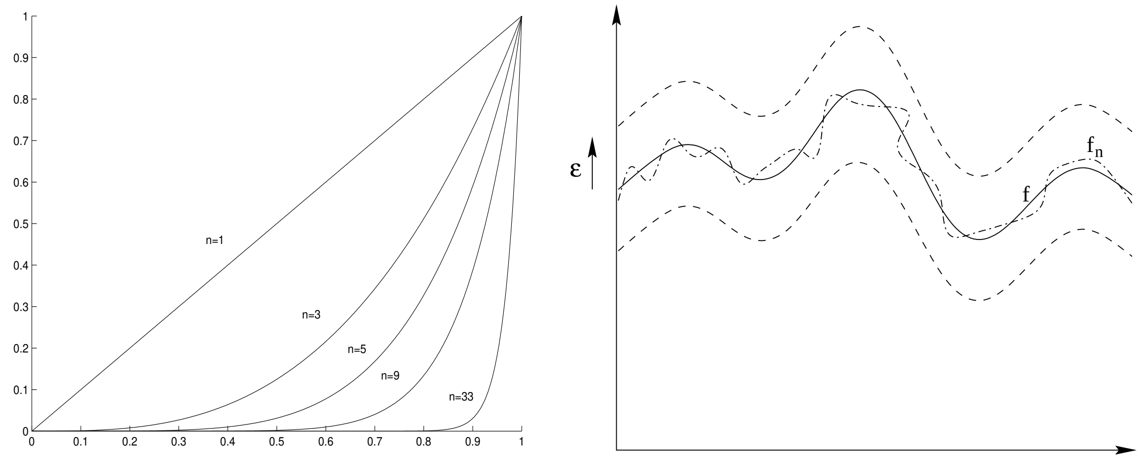

Figure 2.2: Left: the sequence of functions $f_n(x) = x^n$ converges pointwise but not uniformly on $[0, 1]$. Right: graphically, uniform convergence means that for an arbitrarily narrow tubular neighborhood of the limiting function, the functions f_n will be contained in it for all sufficiently large n.

Then, (f_n) converges pointwise to the function δ_1 defined by

$$\delta_1(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases} \tag{2.5}$$

which is clearly discontinuous at $x = 1$.

A natural metric structure on the space of all continuous functions is given in terms of the *uniform* or *supremum* norm defined by:

$$||f|| = \sup_{x \in D} |f(x)| \tag{2.6}$$

This norm is often denoted by $||\cdot||_{\text{sup}}$ or $||\cdot||_\infty$. The origin of the latter notation will become clear when we introduce L^p spaces in Chapter 22. From the definition (2.6) it is clear that a function f is bounded iff $||f||_{\text{sup}} < +\infty$.

Definition 2.8 (uniform convergence) *A sequence of bounded functions (f_n) converges uniformly to a function f iff*

$$\lim_{n\to\infty} ||f_n - f||_{\text{sup}} = 0$$

Note that uniform convergence implies pointwise convergence. The example defined in (2.5) shows that the opposite does not hold. Here is another example.

Example:
$f_n(x) = \tanh nx \to \text{sign}x$, pointwise, but not uniformly.

A crucial property of uniform convergence is that it preserves continuity. Below, we state the theorem, and its proof. Note the strategy of the proof.

Theorem 2.9 *Let (f_n) be a sequence of continuous real-valued functions on a metric space (X, d), and suppose $f_n \to f$ in the uniform norm. Then, f is continuous.*

Proof: In order to show that f is continuous at $x \in X$, we need to prove that for every $\epsilon > 0$, there is a $\delta > 0$ such that for all $y \in X$, with $d(x, y) < \delta$, $|f(x) - f(y)| < \epsilon$.

We start from

$$|f(x) - f(y)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)|$$

First, use the *uniform* convergence to choose an n such that the first and the third term are less than $\epsilon/3$ *for all* $x, y \in X$. Then, use the continuity of f_n at x to find δ such that the middle term is less than $\epsilon/3$ for $d(x, y) < \delta$. The desired result follows. ■

2.3 The spaces $C(K)$

Let K be a compact metric space, and let $C(K)$ denote the set of all continuous functions $f : K \to \mathbb{R}$. Equipped with the metric derived from the uniform norm, $C(K)$ becomes a metric space. Explicitly:

$$d(f, g) = \|f - g\| = \sup_{x \in K} |f(x) - g(x)|$$

for all $f, g \in C(K)$. Note that due to Theorem 2.5, $\|f\| < +\infty$, for all $f \in C(K)$. It is straightforward to verify the properties M1-4 of Definition 1.1.

Theorem 2.10 *$C(K)$, with the uniform norm, is a complete metric space.*

Proof: Let (f_n) be a Cauchy sequence in $C(K)$ for the uniform norm. We have to show that (f_n) converges in the uniform norm. By Theorem 2.9 the limit function f will then automatically be continuous and hence $f \in C(K)$. We do this in two steps a) find a candidate function f for the limit of the sequence, and b) show that the sequence indeed converges uniformly to f.

a) Note that (f_n) converges pointwise due to the completeness of $\mathbb{R}$ and the observations that for all $x \in K$ the sequence of real numbers $(f_n(x))$ is Cauchy. Call the limit of $(f_n(x))$, $f(x)$. This defines f.

b) We show explicitly that $\lim_n \|f_n - f\|_{\text{sup}} = 0$.

$$\|f_n - f\|_{\text{sup}} = \sup_{x \in K} |f_n(x) - f(x)|$$

$$
\begin{aligned}
&= \sup_{x\in K} \lim_{m\to\infty} |f_n(x) - f_m(x)| \\
&= \sup_{x\in K} \liminf_{m\to\infty} |f_n(x) - f_m(x)| \\
&\leq \liminf_{m\to\infty} \sup_{x\in K} |f_n(x) - f_m(x)| \qquad (2.7)
\end{aligned}
$$

For the last step we have used (1.7) The fact that (f_n) is Cauchy in the uniform norm means that, for all $\epsilon > 0$,

$$
\sup_{x\in K} |f_n(x) - f_m(x)| < \epsilon, \text{ for all } n, m \text{ large enough.}
$$

This proves that $\|f_n - f\|_{\sup} \to 0$. ■

$C(K)$ has additional structure: it is also a linear space for the natural operations of addition and multiplication by a scalar of functions, and its metric is derived from a norm. Normed linear spaces that are complete are called *Banach spaces*, and so, $C(K)$ is a Banach space. In fact, one can also multiply two functions (pointwise) and thus give $C(K)$ an *algebra* structure. One says that $C(K)$ is a *Banach algebra*.

Instead of real-valued functions on K, one can equally well consider complex-valued functions and the same result holds: the complex-valued continuous functions on K form a Banach space.

Exercise 2.5 *Let $f_n \in C([a,b])$ be a sequence of functions converging uniformly to a function f. Show that*

$$
\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx \quad ,
$$

i.e., show that $f \mapsto \int_a^b f(x)\,dx$, is continuous on $C([a,b])$.

If X is not compact, continuous functions defined on it may be unbounded. But it is still true that the space of all *bounded continuous functions* on X, denoted by $C_b(X)$, is a Banach space for the supremum norm. In the case of $X = \mathbb{R}$, there is another commonly used Banach space of continuous functions: the space $C_0(\mathbb{R})$ consisting of all continuous functions vanishing at infinity, i.e.,

$$
C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous and } \lim_{x\to\pm\infty} f(x) = 0\}
$$

Exercise 2.6 *Prove that $C_0(\mathbb{R})$ is a closed subspace of $C_b(\mathbb{R})$ with the sup norm.*

Exercise 2.7 *Show that $(C_0(\mathbb{R}), \|\cdot\|_{\sup})$ and $(C_b(\mathbb{R}), \|\cdot\|_{\sup})$, are Banach spaces. (Hint: adapt the proof of Theorem 2.10.)*

Theorem 2.11 *Let K be a compact metric space and let $f : K \to \mathbb{R}$ be a continuous real-valued function on K. Then, f is uniformly continuous.*

Proof: We prove the theorem by contraposition. Suppose f is not uniformly continuous. Then, there exists $\epsilon > 0$ such that for all $\delta > 0$, there are x, y, $d(x, y) < \delta$, but $d(f(x), f(y)) > \epsilon$. We use this for the sequence of δ's, $\delta_n = 1/n$ to obtain two sequences (x_n) and y_n in K with the properties:

$$d(x_n, y_n) < 1/n \quad \text{and} \quad d(f(x_n), f(y_n)) > \epsilon \tag{2.8}$$

Since K is compact there are convergent subsequences of (x_n) and (y_n), which for simplicity we again denote by (x_n) and (y_n), with limits x and y in K, and satisfying (2.8). Clearly, $x = y$, because

$$d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y)$$

and the three terms on the RHS of this inequality converge to 0. We conclude the proof by showing that assuming that all this holds for a continuous function f implies that f is not continuous at x (*reductio ad absurdum*)! Indeed, starting from (2.8), we have

$$\epsilon < d(f(x_n), f(y_n)) \leq d(f(x_n), f(x)) + d(f(x), f(y_n))$$

As the two terms on the RHS converge to) as $m \to \infty$, this contradicts the continuity of f. ■

Let $[a, b] \subset \mathbb{R}$ be a closed interval. Consider the polynomial functions on $[a, b]$, i.e., functions p of the form

$$p(x) = \sum_{k=0}^{n} a_k x^k$$

where the coefficients a_k are arbitrary real numbers and $n \in \mathbb{N}$ is called the degree of p if $a_n \neq 0$.

Our next goal is the classical Weierstrass Approximation Theorem, which states that any continuous function $f : [a, b] \to \mathbb{R}$ can be approximated in the sense of the uniform norm by a polynomial (function), to arbitrary precision. Such an approximation property can be concisely formulated using the notion of dense set.

Theorem 2.12 (Weierstrass Approximation Theorem) *The set of all polynomial functions is dense in $C([a, b])$.*

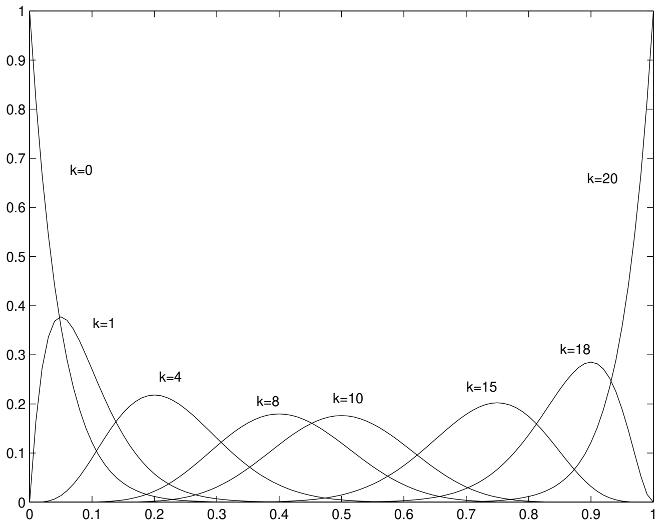

Figure 2.3: The polynominals $x^k(1-x)^{n-k}$, for the case $n = 20$, appearing in the definition of the Bernstein polynomials (2.9).

Proof: We need to show that for any $f \in C([a, b])$ there is a sequence of polynomials p_n such that $\lim_n \|f - p_n\| = 0$.

We first show that it is sufficient to prove the theorem in the case $a = 0, b = 1$. This is so because rescaling and shifting the argument x are norm-preserving transformations from $C([a, b] \to C([0, 1])$ that map the polynomial functions into themselves. More explicitly, define $T : C([a, b] \to C([0, 1])$ by

$$Tf(x) = f(a + (b - a)x)$$

Then, $\|Tf\|_{\text{sup}} = \|f\|_{\text{sup}}$ and T is linear, in particular $T(f - g) = Tf - Tg$, and T is invertible with inverse

$$T^{-1}f = f(\frac{x-a}{b-a})$$

It is also easy to see that t and T^{-1} preserve continuity, and that for any polynomial p, Tp and $T^{-1}p$ are again polynomials. Then, for any $f \in C([a, b])$, $Tf \in C([0, 1])$, and assuming we have shown that the polynomials are dense in $C([0, 1])$, we then have a sequence of polynomials p_n such that $p_n \to Tf$ in $C([0, 1])$ with the uniform norm. From the properties discussed above it follows that the sequence $T^{-1}p_n$ is a sequence of polynomials converging to f.

So, we have to prove that the polynomial functions are dense in $C([0, 1])$. We follow the proof by Bernstein, who gave, for every f, an explicit formula for a sequence of polynomials converging to f. These are called the Bernstein polynomials and are defined as follows:

$$B_n(x;f) = \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k} \tag{2.9}$$

$B_n(x;1) = 1$, by the binomial theorem. Therefore,

$$|f(x) - B_n(x;f)| \leq \sum_{k=0}^{n} |f(x) - f(\frac{k}{n})|\binom{n}{k}x^k(1-x)^{n-k} \tag{2.10}$$

Hence,

$$\begin{aligned} \|B_b(\cdot\,;f) - f\|_{\sup} &= \sup_{0\leq x\leq 1} \sum_{k=0}^{n} \left(f(\frac{k}{n}) - f(x)\right)\binom{n}{k}x^k(1-x)^{n-k} \\ &\leq \sup_{0\leq x\leq 1} \sum_{k=0}^{n} \left|f(\frac{k}{n}) - f(x)\right|\binom{n}{k}x^k(1-x)^{n-k} \end{aligned} \tag{2.11}$$

Next, we use the fact that f is *uniformly* continuous (Theorem 2.11), and pick a $\delta > 0$ such that $|f(k/n) - f(x)| < \epsilon$, for all k such that $|x - (k/n)| < \delta$. And define $I(\delta, x) = \{k||x - n/k| \geq \delta\}$. Then we can estimate the RHS of (2.11) as follows:

$$\begin{aligned} \|B_b(\cdot\,;f) - f\|_{\sup} &\leq \sup_{0\leq x\leq 1} \epsilon \sum_{0\leq k\leq n, k\notin I(\delta,x)} \binom{n}{k}x^k(1-x)^{n-k} \\ &\quad + \sup_{0\leq x\leq 1} \sum_{k\in I(\delta,x)} \left|f(\frac{k}{n}) - f(x)\right|\binom{n}{k}x^k(1-x)^{n-k} \\ &\leq \epsilon\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} \\ &\quad +2\|f\|_{\sup} \sup_{0\leq x\leq 1} \sum_{k\in I(\delta,x)} \binom{n}{k}x^k(1-x)^{n-k} \end{aligned} \tag{2.12}$$

The first term is equal to ϵ, and the second term can be estimated as follows:

$$\begin{aligned} &2\|f\|_{\sup} \sup_{0\leq x\leq 1} \sum_{k\in I(\delta,x)} \binom{n}{k}x^k(1-x)^{n-k} \\ &\leq 2\|f\|_{\sup} \sup_{0\leq x\leq 1} \frac{1}{\delta^2}\sum_{k=0}^{n}(x - \frac{n}{k})^2\binom{n}{k}x^k(1-x)^{n-k} \\ &\leq 2\|f\|_{\sup} \sup_{0\leq x\leq 1} \frac{1}{\delta^2}\sum_{k=0}^{n}(x^2 - 2\frac{k}{n}x + \frac{k^2}{n^2})\binom{n}{k}x^k(1-x)^{n-k} \end{aligned}$$

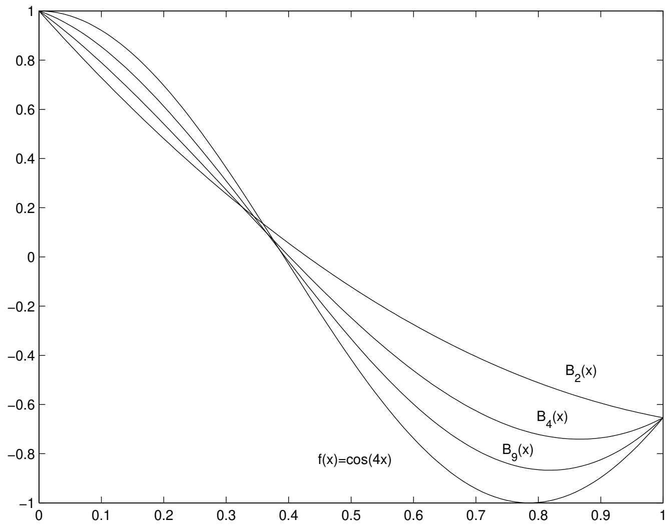

Figure 2.4: Some approximations of the function $x \mapsto \cos(4x)$ by its Berstein polynomials.

$$\leq \frac{2\|f\|_{\sup}}{\delta^2} \sup_{0\leq x\leq 1} \big(x^2 B_n(x;1) - 2xB_n(x;x) + B_n(x;x^2)\big) \tag{2.13}$$

Here, $1, x$, and x^2 denote the corresponding polynomial functions, when they serve as the second argument of B_n. We can now compute $B_n(x;1), B_n(x;x)$, and $B_n(x;x^2)$ explicitly to finish the estimate.

$$\begin{array}{rcl} B_n(x;1) = & 1 & \text{for all } n \geq 0 \\ B_n(x;x) = & x & \text{for all } n \geq 1 \\ B_n(x;x^2) = & \frac{n-1}{n}x^2 + \frac{1}{n}x & \text{for all } n \geq 1 \end{array} \tag{2.14}$$

All combined we obtain the estimate

$$\|B_n(\cdot\,; f) - f\|_{\sup} \leq \epsilon + \frac{\|f\|_{\sup}}{2n\delta^2}$$

which concludes the proof.

[Hint: prove (2.14) using the binomial theorem to expand $(x+y)^n$ and take derivatives with respect to x to obtain the identities:

$$\begin{aligned} x(x+y)^{n-1} &= \sum_{k=0}^{n} \binom{n}{k} \frac{k}{n} x^k y^{n-k} \\ \frac{1}{n}x(x+y)^{n-1} + \frac{n-1}{n}x^2(x+y)^{n-2} &= \sum_{k=0}^{n} \binom{n}{k} \frac{k^2}{n^2} x^k y^{n-k} \quad] \end{aligned} \tag{2.15}$$

■

Note that the proof works for both real-valued and complex-valued functions.

The space of all continuous functions (with values in $\mathbb{R}$ or $\mathbb{C}$) defined on the unit circle, T, is closely related to $C([0,1])$. A very usefull density result for $C(T)$ is the following.

If we parametrize the unit circle by an angle θ, i.e.,

$$T = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$$

any continuous function $f : T \to \mathbb{C}$ can be considered as a continuous periodic function of the real variable θ. E.g., the trigonometric polynomials are continuous. A *trigonometric polynomial* of degree n is a function of the form

$$P_n(\theta) = \sum_{k=-n}^{n} a_k e^{ik\theta}$$

the a_k are arbitrary complex coefficients.

The following density result, which has many applications, will be proved later in Chapter 8, when we discuss Fourier series.

Theorem 2.13 *The set of all trigonometric polynomials is dense in $C(T)$ with the uniform norm.*

2.4 Compact subsets of $C(K)$

Recall that the proof of the Heine-Borel Theorem used the finite-dimensionality of $\mathbb{R}^n$ in a crucial way. In infinite-dimensional spaces compact sets are also necessarily bounded and closed, but these properties are no longer sufficient. The aim of this subsection is to prove a characterization of the compact subsets of $C(K)$, the space of all real-valued continuous functions on a compact space K with the uniform norm (the Arzelá-Ascoli Theorem). For this we will need the fact that every compact set has the *finite-open-subcover* property. First, we explain what the finite-open-subcover property is.

A family of open sets $G_\alpha \subset X$ is called an *open cover* of $A \subset X$ if

$$A \subset \bigcup_\alpha G_\alpha$$

The index α runs through an arbitrary index set that need not be countable. For any subset A of a metric space (X, d), it is obvious that for any $\epsilon > 0$, A can be covered with a family of open balls in X of radius ϵ, meaning that there exists a family of points $x_\alpha \in X$ such that

$$A \subset \bigcup_\alpha B_\epsilon(x_\alpha) \tag{2.16}$$

Indeed, it is always possible to take $\{x_\alpha\} = A$. Let us call such a cover an *ϵ-cover* of A. A subset $\{x_i\} \subset A$ with the property (2.16) is called an *ϵ-net* for A, and a finite ϵ-net if the set of points is finite. $A \subset X$ is called *totally bounded* if A has a finite ϵ-net for every $\epsilon > 0$.

We say that A has the finite-open-subcover property if for all open covers $\{G_\alpha\}$ of A there is a finite subcover. By this we mean that we can find a finite subset of the G_α, say $\{G_{\alpha_1}, \ldots, G_{\alpha_k}\}$ such that

$$A \subset \bigcup_{i=1}^{k} G_{\alpha_i}$$

The finite-open-subcover property is equivalent to compactness. We will prove one direction of this equivalence in Theorem 2.16. First, we prove some preliminary results.

Theorem 2.14 *Every compact metric space K is totally bounded, i.e., has a finite ϵ-net for every $\epsilon > 0$.*

Proof: Let $\epsilon > 0$, and suppose that K has no finite ϵ-net. Then, for every finite subset $\{x_1, \ldots, x_n\} \subset K$, there exists

$$x_{n+1} \in K \setminus \bigcup_{i=1}^{n} B_\epsilon(x_i)$$

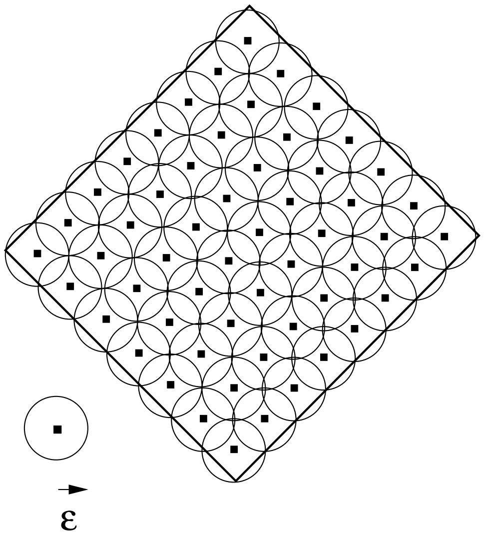

Figure 2.5: A set with an ϵ-net for it.

So, we can find an infinite sequence (x_n) in K with the property that $d(x_n, x_m) > \epsilon$ for all $n \neq m$. Clearly, this sequence does not contain any Cauchy subsequence, and hence no convergent subsequence. We conclude that K must have a finite ϵ-net for every $\epsilon > 0$. ■

A metric space is called *separable* if it contains a countable dense subset.

Corollary 2.15 *Every compact metric space K is separable, i.e., contains a countable dense subset.*

Proof: For every $n \geq 1$, let A_n be a finite $\frac{1}{n}$-net of K. Such A_n exist due to Theorem 2.14. Then, $A := \bigcup_{n \in \mathbb{N}} A_n$ is dense in K, and, as a countable union of finite sets, A is countable. ■

The notation LHS := RHS is often used when the RHS *defines* the LHS. Other commonly used equivalent notations are $\equiv$ and $\stackrel{\text{def}}{=}$.

Theorem 2.16 *Let (X, d) be a metric space and suppose $K \subset X$ is compact. Then K has the finite-open-subcover property.*

Proof: We will first show that any open cover $\{G_\alpha\}$ of K contains a countable subcover. From Corollary 2.15, we know that K contains a countable dense subset A. Consider the family $\mathcal{B}$ of open balls with center in A and radius in $\mathbb{Q}$. Then, $\mathcal{B}$ is a countable family. Consider the subset $\mathcal{C}$ of $\mathcal{B}$ consisting of all $B \in \mathcal{B}$ that are contained in one of the G_α. As a subset of a countable set, $\mathcal{C}$ is still countable. For every $B \in C$, pick α_B such that $B \subset G_\alpha$. Then, $\{G_{\alpha_B}\}_{B\in\mathcal{C}}$ is a countable subset of $\{G_\alpha\}$, and that it is a cover of K can be seen as follows. For every $x \in K$, there is a member of the open cover G_α, such that $x \in G_\alpha$. As G_α is open, there is an open ball $B_\epsilon(x) \subset G_\alpha$, and we can assume that ϵ is rational. As A is dense, there exists $y \in A$ such that $d(x, y) < \epsilon/2$. Then $x \in B_{\epsilon/2}(y) \subset G_\alpha$, and, hence, $B_{\epsilon/2}(y) \in \mathcal{C}$, and it follows that $\mathcal{C}$ is a cover. By construction we have

$$\bigcup_{B\in\mathcal{C}} B \subset \bigcup_{B\in\mathcal{C}} G_{\alpha_B}$$

Therefore, $\{G_{\alpha_B}\}_{B\in\mathcal{C}}$ is a countable subcover of $\{G_\alpha\}$.

Next, we show that any countable open cover $\{G_i \mid i \in \mathbb{N}\}$ contains a finite subcover. To prove this we will perform a construction that either selects a finite subcover or yields a sequence (x_n) of elements $x_n \in K$ without a convergent subsequence with limit in K. If K is compact the last possibility must be excluded and hence there must be a finite subcover.

Define

$$R_n = K \setminus \bigcup_{i=1}^{n} G_i$$

If $R_n = \emptyset$ for some n, we have found a finite subcover and we are done. If $R_n \neq \emptyset$, for all n, we can construct a sequence $y_k \in K$ as follows. Pick $y_1 \in R_1$. Let n_1 be such that $y_1 \in G_{n_1}$. Then pick $y_2 \in R_{n_1}$, and so on. The result is a sequence y_k such that

$$y_{k+1} \notin \bigcup_{i=1}^{n_k} G_i \quad \text{and} \quad y_{k+1} \in \bigcup_{i=1}^{n_{k+1}} G_i \tag{2.17}$$

Suppose (y_{k_n}) is a convergent subsequence of (y_k), with limit y. Then, $y \notin G_i$, for any i, for if $y \in G_i$ for some i, (y_{k_n}) cannot converge to y. This follows from (2.17) and the fact that the G_i are open. We conclude that $y \notin K \subset \bigcup_i G_i$.

As K is assumed to be compact the possibility that all $R_n \neq \emptyset$ must be excluded and hence there is a finite subcover of K. ■

Exercise 2.8 (Dini's Monotone Convergence Theorem)
Suppose $f_n \in C([0,1])$ converges pointwise and monotone decreasingly *to $f \in C([0,1])$. I.e., for all $x \in [0,1]$, the sequence $(f_n(x))$ is monotone decreasing and converges to $f(x)$. Prove that f_n converges uniformly to f.*

For the characterization of the compact subsets of $C(K)$ we will need the notion of *equicontinuity*.

Definition 2.17 (Equicontinuity) *Let F be a family of continous functions from (X, d) to (Y, d). F is equicontinuous iff for all $x \in X$, and for all $\epsilon > 0$, there is a $\delta > 0$ such that for all y with $d(x, y) < \delta$, one has $d(f(x), f(y)) < \epsilon$, for all $f \in F$.*

The crucial point in this definition is that δ can be chosen independent of f, while it may depend on x. If δ can be chosen independent of x as well, the family is *uniformly equicontinuous*. If X is compact equicontinuity implies uniform equicontinuity, as is shown in the following theorem.

Theorem 2.18 *Let F be a family of continuous functions from (K, d) to (Y, d), where K is compact. If F is equicontinuous, then F is uniformly equicontinuous.*

Proof: Suppose the equicontinuous family F is not uniformly equicontinuous. Then, there must be $\epsilon > 0$, such that for all $\delta > 0$ there are x_δ, and f_δ, such that there exist $y, d(x_\delta, y) < \delta$, but $d(f_\delta(y), f_\delta(x_\delta)) > 2\epsilon$.

In particular there are sequences x_n, y_n, and f_n, such that for all n, $d(x_n, y_n) < 1/n$, and

$$d(f_n(y_n), f_n(x_n)) > 2\epsilon \tag{2.18}$$

As K is compact, (x_n) has a convergent subsequence. Let such a subsequence again be denoted by (x_n). We then have $x_n \to x_0$, and hence $y_n \to 0$. Choose n large enough such that $d(x_n, x_0) < \delta$ and $d(y_n, x_0) < \delta$. Because of (2.18) we must have at least one of the following two: $d(f_n(y_n), f_n(x_0)) > \epsilon$ or $d(f_n(x_n), f_n(x_0)) > \epsilon$. As $\delta > 0$ is arbitrary, this is in contradiction with F being equicontinuous in x_0. ■

Note that this theorem implies Theorem 2.11. It suffices to take $F = \{f\}$.

A set A is called *relatively compact*, or *precompact*, if its closure $\overline{A}$ is compact.

Theorem 2.19 (Arzelà-Ascoli) *Let K be compact. A set $A \subset C(K)$ with the uniform norm is compact iff A has the following two properties:*
i) A is closed and bounded, and
ii)A is equicontinuous.
Equivalently, A is precompact iff it satisfies:
i′) A is bounded, and
ii′) A is equicontinuous.

Proof: We divide the proof into four parts:
1) precompact $\Rightarrow$ bounded
2) compact $\Rightarrow$ closed
3) precompact $\Rightarrow$ equicontinuous
4) bounded + equicontinuous $\Rightarrow$ precompact.

1) If A is not bounded there is a sequence of functions $f_n \in A$, satisfying $\|f_{n+1}\| \geq \|f_n\| + 1$. Then, $\|f_n - f_m\| \geq 1$, for all $n \neq m$. It follows that (f_n) that does not have a Cauchy subsequence, and hence does not have a convergent subsequence.

2) If A is not closed, there is a sequence $f_n \in A$ converging to a limit $f \in \overline{A} \setminus A$. All subsequences of this sequence also converge to f, and hence do not have a limit *in* A.

3) If A is precompact, then $\overline{A}$ is compact by definition. Fix $\epsilon > 0$. A is dense in $\overline{A}$. Therefore

$$\overline{A} \subset \bigcup_{f \in A} B_\epsilon(f)$$

By Theorems 2.14 and 2.16 there is then a finite subset $\{f_1, \ldots, f_k\} \subset A$ such that

$$\overline{A} \subset \bigcup_{i=1}^{k} B_\epsilon(f_i)$$

As each f_i is uniformly continuous (Theorem 2.11), there exist $\delta_i > 0$ such that for all $x, y \in K$, we have

$$d(x,y) < \delta_i \Rightarrow |f_i(x) - f_i(y)| < \epsilon. \tag{2.19}$$

Then, for any $f \in A$, there is an i such that $\|f - f_i\| < \epsilon$ (because $\{f_i\}$ is an ϵ-net). Take $\delta = \min_i \delta_i$. We conclude that for $d(x,y) < \delta$

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \\ &\leq 3\epsilon \end{aligned} \tag{2.20}$$

As ϵ is arbitrary, and δ is independent of f, this shows that A is equicontinuous.

4) We will show, using boundedness and equicontinuity, that every sequence in A has a convergent subsequence (then, necessarily, with a limit in $\overline{A}$). By Corollary 2.15 there is a countable dense set $\{x_1, x_2, x_3, \ldots\} \subset K$. We now choose a first subsequence $(f_{1,n})$ of (f_n) such that the sequence $(f_{1,n}(x_1))$ converges in $\mathbb{R}$. Such a subsequence exists because A, and hence $(f_n(x_1))$, is bounded. Next, we choose a subsequence $(f_{2,n})$ of $(f_{1,n})$ such that $(f_{2,n}(x_2))$ converges, which exists for the same reason. And so on. $(f_{(k+1),n})$ is a subsequence of $(f_{k,n})$ such that $(f_{(k+1),n}(x_{k+1}))$ converges as $n \to \infty$. Finally, the subsequence we are after is the "diagonal" of the "table" of functions $f_{i,j}$, i.e., define the subsequence (g_k) of (f_n) by $g_k = f_{k,k}$.

By construction (g_k) has the property that $g_k(x_i)$ converges in $\mathbb{R}$ as $k \to \infty$ for all x_i in a dense subset of K. So far, we only used the boundedness of A. The equicontinuity is needed to ensure that (g_k) converges uniformly to a limit function g.

Let $\epsilon > 0$. Since A is equicontinuous, A is uniformly equicontinuous by Theorem 2.18. Therefore, and since all $g_k \in A$, there exists $\delta > 0$ such that for all x, y such that $d(x,y) < \delta$,

$$|g_k(x) - g_k(y)| < \epsilon$$

Since $\{x_i\}$ is dense in K we have

$$K \subset \bigcup_{i=1}^{\infty} B_\delta(x_i)$$

By Theorem 2.16 there is a finite subset $\{x_{i_1}, \dots, x_{i_l}\}$ such that

$$K \subset \bigcup_{j=1}^{l} B_\delta(x_{i_j})$$

Since for all $j = 1, \dots, l$, $(g_k(x_{i_j}))$ is convergent, and hence Cauchy, we can now find N such that

$$\text{for all } p, q \geq N, \quad |g_p(x_{i_j}) - g_q(x_{i_j})| < \epsilon \text{ for all } j = 1, \dots, l$$

Now, for any $x \in K$,

$$\begin{aligned} |g_p(x) - g_q(x)| &\leq |g_p(x) - g_p(x_{i_j})| + |g_p(x_{i_j}) - g_q(x_{i_j})| + |g_q(x_{i_j}) - g_q(x)| \\ &\leq 3\epsilon \end{aligned} \tag{2.21}$$

where we have made use of the δ-cover to choose x_{i_j} appropriately. It follows that (g_k) is a Cauchy sequence for the uniform norm. As $C(K)$ is complete (Theorem 2.10), it converges. ■

Remarks:

1. In the proof of 4) we used Cantor's "*diagonal argument*". Keep it in mind! It is used in other proofs. Also note the repeated use of the $\epsilon/3$-trick (in the disguise of a 3ϵ-trick).

2. The same result can be proved (in exactly the same way) for a more general situation: consider the space $C(K, Y)$ of all continuous functions $f : K \to Y$, with K a compact and Y an arbitrary metric space. Define a metric on $C(K, Y)$ by
$$d(f, g) = \sup_{x \in K} d(f(x), g(x))$$
Then $(C(K, Y), d)$ is a complete metric space and the compact subsets of $C(K, Y)$ are exactly those that have the properties i) and ii) stated in the Arzelà-Ascoli Theorem.

3. Parts 1) and 2) of the proof work as stated for arbitrary normed spaces.

Examples and counterexamples

We start with two examples of bounded sets in $C([0, 1])$ that are not precompact (and hence their closures are bounded and closed but not compact). Both examples make use of the fact that, as a linear space, $C([0, 1])$ is infinite-dimensional.

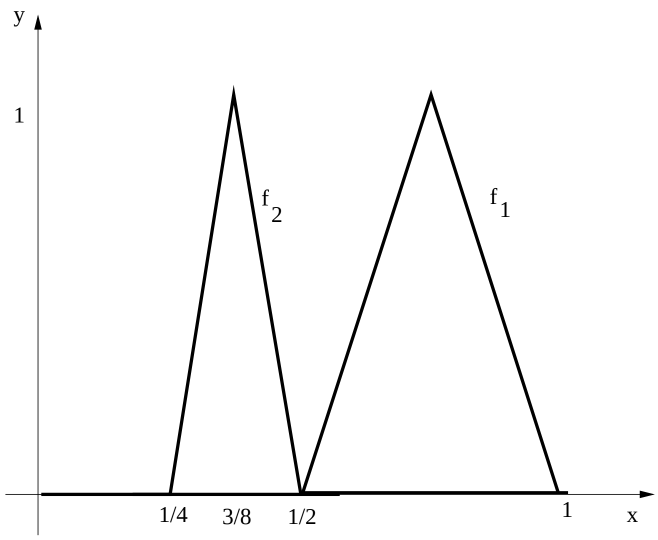

Figure 2.6: The sequence of bounded continuous functions defined in Example 1 is not equicontinuous.

1. For each $n \geq 0$ define a function $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 2^{-(n+1)} \\ 2^{n+2}(x - 2^{-(n+1)}) & \text{if } 2^{-(n+1)} \leq x \leq 3 \cdot 2^{-(n+2)} \\ 2^{n+2}(2^{-n} - x) & \text{if } 3 \cdot 2^{-(n+2)} \leq x \leq 2^{-n} \end{cases} \tag{2.22}$$

Then $\|f_n\|_\infty = 1$, for all $n \geq 1$, and so $\{f_n\}$ is bounded, but $\|f_n - f_m\|_\infty = 1$ for all $n \neq m$. Hence, the sequence (f_n) does not have any Cauchy subsequences, and therefore, as a set it can not be precompact. Look at the graphs of the functions f_n (Figure 2.6) and show explicitly that $\{f_n\}$ is not equicontinuous.
2. The same phenomenon occurs for the set of functions f_n defined on $[0,1]$ by $f_n(x) = \sin(n\pi x)$.

For a subset of an infinite-dimensional linear space to be (pre)compact "most of it" must be contained in a finite-dimensional subspace. Without making this more precise, we illustrate this with the following example.

3. Define a subset A of $C([0,1])$ as follows:

$$A = \{f \in C([0,1]) \mid f(x) = \textstyle\sum_{n=1}^{\infty} a_n \sin(n\pi x) \text{ with } a_n \in \mathbb{R}, \text{ and } \sum_{n=0}^{\infty} n^2 a_n^2 \leq 1\}$$

It can be shown that is precompact in $(C([0,1]), \|\cdot\|_\infty)$, by verifying the assumptions of the Arzelà-Ascoli Theorem.

Next, we give two examples of useful sufficient conditions for equicontinuity.

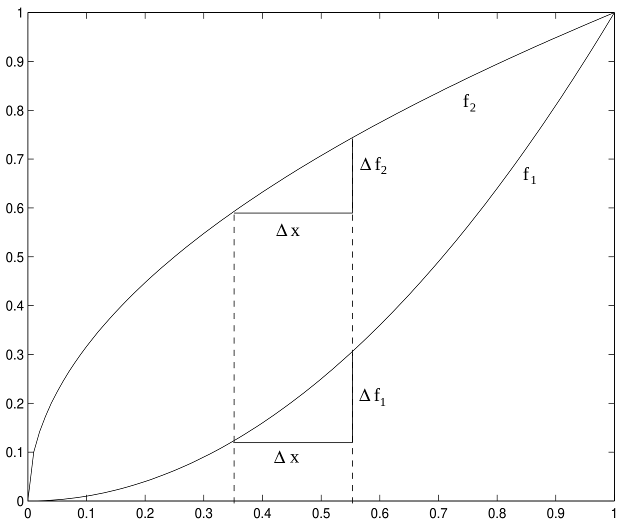

Figure 2.7: The graph of two continuous functions on $[0,1]$: $f_1(x) = x^2$, and $f_2(x) = \sqrt{x}$. f_1 is Lipschitz on $[0,1]$, but f_2 is not Lipschitz in the point 0. This follows from the observation that the ratio $\Delta f_1/\Delta x$, is bounded (for arbitrarily small Δx, and everywhere in $[0,1]$), while $\Delta f_2/\Delta x$ is unbounded for small Δx near 0.

4. Let (X,d) be a metric space. A function $f : X \to \mathbb{R}$ is called (globally) *Lipschitz continuous* if there exists a constant $C \geq 0$ such that

$$|f(x) - f(y)| \leq Cd(x,y)$$

It is obvious that Lipschitz continuous $\Rightarrow$ continuous. For a Lipschitz continuous f we can define the *Lipschitz constant* $\mathrm{Lip}(f)$ of f by

$$\mathrm{Lip}(f) = \sup_{x \neq y \in X} \frac{|f(x) - f(y)|}{d(x,y)}$$

or, equivalently

$$\mathrm{Lip}(f) = \inf\{C \geq 0 \mid |f(x) - f(y)| \leq Cd(x,y), \text{ for all } x,y \in X\}$$

For any compact metric space K and any constant M define

$$A = \{f \in C(K) \mid \mathrm{Lip}(f) \leq M\} \tag{2.23}$$

Then, A is equicontinuous. Indeed, for all $\delta > 0$ we have

$$d(x,y) < \delta \Rightarrow |f(x) - f(y)| \leq M\delta \quad \text{for all } f \in A; x,y \in K$$

The set A defined by (2.23) is not bounded. E.g., it is easy to see that $f \in A \Rightarrow f + a \in A$, for any $a \in \mathbb{R}$, and hence, A is unbounded. Here is a bounded set $B \subset A$, in the case $K = [0,1]$:

$$B = \{f \in A \mid f(0) = 0\}$$

Now, B is bounded because

$$|f(x)| = |f(x) - f(0)| \leq M|x - 0| \leq M$$

In fact, B is also closed (and hence compact). In fact, A is closed for arbitrary K and M. To check this we use once more our favorite relation involving lim inf. Let f_n be a sequence in B converging uniformly to f. Then

$$\begin{aligned} \mathrm{Lip}(f) &= \sup_{x \neq y \in K} \frac{|f(x) - f(y)|}{d(x,y)} \\ &= \sup_{x \neq y \in K} \liminf_{n \to \infty} \frac{|f_n(x) - f_n(y)|}{d(x,y)} \\ &\leq \liminf_{n \to \infty} \sup_{x \neq y \in K} \frac{|f_n(x) - f_n(y)|}{d(x,y)} \\ &\leq \liminf_{n \to \infty} M = M \end{aligned}$$

5. Let $C^1([0,1])$ denote the space of all continuously differentiable real-valued functions on $[0,1]$, and suppose $f \in C^1([0,1])$. Then, by the mean-value theorem, for every $x, y \in [0,1]$, $x < y$, there is z, $x < z < y$, such that

$$f(x) - f(y) = f'(z)(x - y)$$

It follows that $\mathrm{Lip}(f) \leq \|f'\|_\infty$, which is finite because $f' \in C([0,1])$ by assumption. For any constants M and N define a set $D \subset C^1([0,1])$ by

$$D = \{f \in C^1([0,1]) \mid \|f\|_\infty \leq M, \|f'\|_\infty \leq N\}$$

It follows that D is bounded and equicontinuous, and hence precompact in $C([0,1])$ with the uniform norm.

2.5 Existence of local solutions of ODE's

This section is devoted to a simple application of the Arzelà-Ascoli Theorem. Consider a scalar ordinary differential equation (ODE) of the type

$$y' = F(x, y)$$

on an open domain $D \subset \mathbb{R}$, and with "initial" condition $y(x_0) = y_0$. Solving this ODE on $D_0 \subset D$ means finding a function (or functions) $f : D_0 \to \mathbb{R}$ such that f and its derivative f' satisfy

$$\left.\begin{array}{rcl} f'(x) & = & F(x, f(x)) \quad \text{for all } x \in D_0 \\ f(x_0) & = & y_0 \end{array}\right\} \tag{2.24}$$

We made a distinction between D and D_0, because it is a priori not clear one can find a solution f defined on the whole of D on which the problem is posed. But under mild conditions on F there is always an open interval D_0, containing x_0, on which at least one solution exists.

Theorem 2.20 (Local existence for scalar ODE) *If F is continuous on an open set $R \in \mathbb{R}^2$, containing (x_0, y_0), then there exists an open interval $D_0 \subset \mathbb{R}$, containing x_0, and a function $f : D_0 \to \mathbb{R}$ solving the system (2.24).*

Proof: The proof is mainly constructive, in the following sense. We will first construct a sequence of functions that "approximately" solve the ODE. This sequence will consist of elements in a precompact subset (Arzelà-Ascoli) of some $C([a, b])$, and hence it contains a uniformly convergent subsequence. We then show that any limit of such a subsequence must solve (2.24).

For any $\epsilon > 0$ we will define an approximate solution f_ϵ in the sense that

$$|f'_\epsilon(x) - F(x, f_\epsilon(x))| \leq \epsilon \,, \quad f_\epsilon(x_0) = y_0$$

for all x in some interval $[x_0 - d, x_0 + d]$, except for a finite number of points (where f'_ϵ does not exist). The f_ϵ are piecewise linear continuous functions obtained by the following, rather naive, construction:

$$f_\epsilon(x) = a_k + b_k(x - (x_0 + k\delta)) \,, \qquad \text{for } x_0 + k\delta \leq x \leq x_0 + (k+1)\delta$$

for some appropriate $\delta > 0$, and a similar expression for $x \leq x_0$. The parameters a_k and b_k are recursively defined by

$$\begin{array}{rcl} a_0 & = & y_0 \\ b_0 & = & F(x_0, y_0) \\ a_k & = & a_{k-1} + b_{k-1}\delta \\ b_k & = & F(x_0 + k\delta, a_k) \end{array}$$

for $k \geq 1$. Then $f'_\epsilon(x) = b_k$, for $x_0 + k\delta < x < x_0 + (k+1)\delta$, and therefore we have

$$\begin{array}{rcl} |f'_\epsilon(x) - F(x, f_\epsilon(x))| & = & |b_k - F(x, a_k + b_k(x - (x_0 + k\delta))| \\ & = & |F(x_0 + k\delta, a_k) - F(x, a_k + b_k(x - (x_0 + k\delta))| \end{array} \tag{2.25}$$

for $x_0 + k\delta \leq x \leq x_0 + (k+1)\delta$, and

$$|x_0 + k\delta - x| \leq \delta \tag{2.26}$$
$$|a_k - a_k + b_k(x - (x_0 + k\delta))| \leq b_k\delta \tag{2.27}$$

Let $R \subset D \subset \mathbb{R}^2$ be a closed rectangle with center (x_0, y_0). We will choose R such that the graph of each f_ϵ, for $x \in [x_0 - d, x_0 + d]$ is contained in R (See Figure 2.8). Then

$$|b_k| \leq \sup_{(x,y)\in R} |F(x,y)| \leq M$$

for some $M \in \mathbb{R}$, because F is continuous. In fact, R is compact, and hence, F restricted to R is uniformly continuous. It then follows form (2.26) and (2.27) that for every ϵ, there exist $\delta > 0$ such that

$$|F(x_0 + k\delta, a_k) - F(x, a_k + b_k(x - (x_0 + k\delta))| \leq \epsilon$$

Combined with (2.25) this shows that f_ϵ is an approximate solution.

A suitable rectangle R is found as follows. First, consider a rectangle $R_1 \subset D$, with center (x_0, y_0). Define

$$M = \sup_{(x,y)\in R_1} |F(x,y)|$$

Then choose $R \subset R_1$ of the form $[x_0 - d, x_0 + d] \times [y_0 - dM, y_0 + dM]$. Figure 2.8 makes it crystal clear why this choice of R works.

Let us summarize the properties of the approximate solution f_ϵ: for each ϵ, δ is found as described above. Then f_ϵ is piecewise linear on intervals of length δ. The slope of each linear piece is $\leq M$, and, therefore, the graph of f_ϵ lies in the cone bounded by $x_0 + Mx$ and $x_0 - Mx$ (see Figure 2.8). This also means that each f_ϵ is Lipschitz continuous with $\mathrm{Lip}(f_\epsilon) \leq M$. By construction we also have $f_\epsilon(x_0) = y_0$, for all ϵ. From example 4 in Section 2.4 we then know that the set $\{f_\epsilon \mid 0 < \epsilon \leq Md\}$ is bounded and equicontinuous, and hence, by the theorem of Arzelà-Ascoli (Theorem 2.19), precompact. It follows that there is a sequence $\epsilon_n \downarrow 0$ such that $f_{\epsilon_n} \to f$ uniformly on $[x_0 - d, x_0 + d]$.

It remains to be shown that this limiting function f solves (2.24). This goes as follows. As f_ϵ is piecewise linear we have

$$\begin{aligned} f_\epsilon(x) &= f_\epsilon(x_0) + \int_{x_0}^{x} f'_\epsilon(s)\,ds \\ &= y_0 + \int_{x_0}^{x} F(s, f_\epsilon(s))\,ds + \int_{x_0}^{x} [f'_\epsilon - F(s, f_\epsilon(s))]\,ds \end{aligned}$$

where f'_ϵ is not necessarily defined in points of the form $x_0 \pm k\delta$, but this does not affect the value of the integral (e.g., one could set the derivative equal to 0 in those

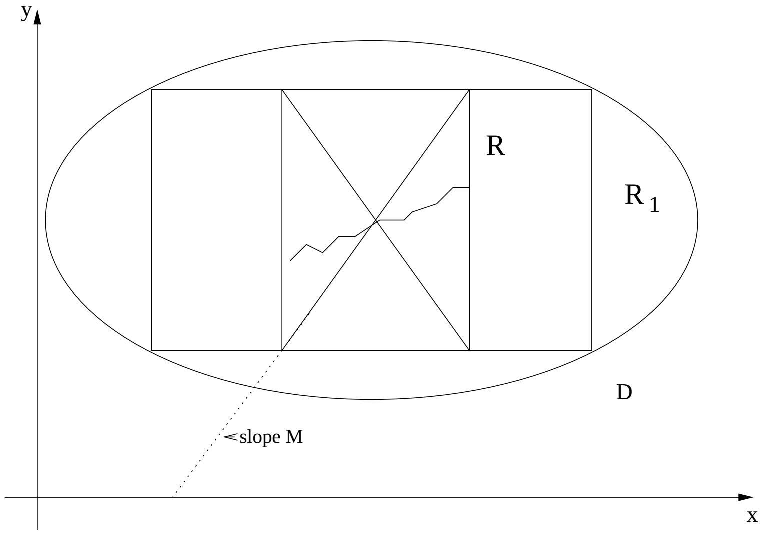

Figure 2.8: The construction of the rectangle R used in the proof of Theorem 2.20.

points). Now, Exercises 2.4 and 2.5 come in handy to show that

$$\begin{aligned} f(x) &= \lim_{n\to\infty} \left\{ y_0 + \int_{x_0}^{x} F(s, f_{\epsilon_n}(s))\, ds + \int_{x_0}^{x} \left[f'_{\epsilon_n} - F(s, f_{\epsilon_n}(s)) \right] ds \right\} \\ &= y_0 + \int_{x_0}^{x} F(s, f(s))\, ds \qquad (2.28) \end{aligned}$$

Using the Fundamental Theorem of Calculus we conclude that the RHS of (2.28) is differentiable and therefore also the LHS is, and

$$f'(x) = F(x, f(x))$$

■

We observe that we also proved that there exists a differentiable solution.

Note that uniqueness of the solution is not part of Theorem 2.20. In fact, there are continuous F for which (2.24) has more than one solution on any open interval around x_0, as is shown in the following examples.

Examples of non-uniqueness:

1. Consider (2.24) with $F(x, y) = \sqrt{y}, y \geq 0$, and $x_0 = y_0 = 0$. I.e., we look for solutions of

$$\begin{aligned} f'(x) &= \sqrt{f(x)} \qquad (2.29) \\ f(0) &= 0 \qquad (2.30) \end{aligned}$$

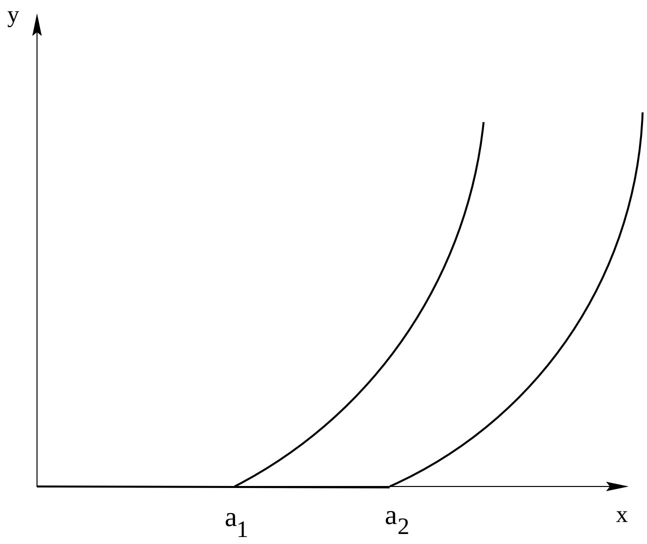

Figure 2.9: Two solutions of the ODE of Example 1.

Clearly, $f \equiv 0$ is a solution, but there is also the following one-parameter family of non-zero solutions (See Figure 2.9):

$$f(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{1}{4}(x-a)^2 & \text{if } x > a \end{cases}$$

for any $a \geq 0$. The construction used in the proof of Theorem 2.20 would only yield the zero solution. One could slightly modify it to produce other solutions as well. In any case this example illustrates clearly that under the conditions stated in Theorem 2.20, one does not necesarily specify a unique function by the system (2.24).

To get a clue as to what extra condition would be needed to guarantee uniqueness of the solution, consider the following example, which is a generalization of Example 1.

2. For $\alpha \geq 0$, we consider

$$\left.\begin{array}{rcl} f'(x) & = & |f(x)|^\alpha \\ f(0) & = & 0 \end{array}\right\} \tag{2.31}$$

Again there is always the solution $f \equiv 0$. When one tries to find other solutions of the form

$$f(x) = \begin{cases} 0 & \text{if } x \leq a \\ C(x-a)^\nu & \text{if } x > a \end{cases}$$

one arrives at the following condition on the parameters C and ν:

$$\alpha = 1 - \frac{1}{\nu} \quad , c^{1-\frac{1}{\nu}} = C\nu \tag{2.32}$$

For $f \not\equiv 0$, we need $C \neq 0$, and the initial condition $f(0) = 0$ is only satisfied if $\nu > 0$. From (2.32) we conclude then conclude that $\alpha < 1$. If $\alpha \geq 1$ there are no non-zero solutions of the proposed from. In fact, as a direct application of the following proposition, one can show that (2.31) has indeed only one solution when $\alpha \geq 1$.

Proposition 2.21 *If F is continuous on an open set $R \subset \mathbb{R}^2$, containing (x_0, y_0), and F is Lipschitz continuous in its second argument, with a Lipschitz constant independent of x (on the domain R). Then, on any interval containing x_0 there exists at most one solution of the system (2.24).*

Proof: The strategy of this uniqueness proof is a very common one: suppose there are two solutions, say f and g, show that the difference $\Delta := f - g$ satisfies an equation that implies $\Delta = 0$.

If f and g both satisfy (2.24) on a interval I containing x_0, it is easy to see that Δ must satisfy

$$\left.\begin{array}{rcl} \Delta'(x) & = & F(x, f(x)) - F(x, g(x)) \quad , \text{ for all } x \in I \\ \Delta(x_0) & = & 0 \end{array}\right\} \tag{2.33}$$

As F is Lipschitz continuous in its second argument, there is a constant C such that

$$|F(x, f(x)) - F(x, g(x))| \leq C|f(x) - g(x)| = C|\Delta(x)|$$

for all $x \in I$. This implies that the function G defined by

$$G(x) = \frac{F(x, f(x)) - F(x, g(x))}{\Delta(x)}$$

is bounded on I. On the other hand we have

$$\frac{\Delta'(x)}{\Delta(x)} = G(x)$$

for all $x \in I, x \neq x_0$. This equation can be integrated to obtain

$$\frac{\Delta(y)}{\Delta(x)} = \exp\{\int_x^y G(s)\, ds\}$$

for all $x \neq x_0$. Using the boundedness of G we conclude that Δ, which is by assumption a differentiable and hence continuous function, satisfies

$$e^{-M}|\Delta(x)| \leq |\Delta(y)| \leq e^M|\Delta(x)|$$

for all $x, y \in I$, and where M can be taken to be $C|I|$, with $|I|$ denoting the length of the interval I. Hence, Δ either never vanishes or is identically 0. As $\Delta(x_0) = 0$, we must have $\Delta = 0$. ■

We will see another, slightly more general proof of Proposition 2.21 in Section 5 as an application of the Contraction Mapping Theorem.