23 Sobolev Spaces

Many problems in applied analysis involve differentiable functions. Roughly speaking, *Sobolev spaces* are Banach spaces of functions with derivatives in some L^p space. We encountered a special case in the chapter on Fourier series. Here, we give the general definition and highlight some of the main results.

23.1 Definitions

In this section we will consider $\mathbb{R}^d$ with the d-dimensional Lebesgue measure. Let Ω be a measurable subset of $\mathbb{R}^d$. If $\Omega_0 \subset \Omega$, and $f : \Omega \to \mathbb{C}$, we will use the same symbol f to denote its restriction to Ω_0. Therefore, it makes sense to ask whether an element $f \in L^p(\Omega)$, is in $L^q(\Omega_0)$ or not, for any $1 \leq p, q \leq \infty$, and measurable $\Omega_0 \subset \Omega$.

It is useful to introduce *local* L^p spaces: a measurable function f is in $L^p_{\text{loc}}(\Omega)$ if $f \in L^p(K)$, for all *compact* subsets $K \subset \Omega$. Clearly, $L^p \subset L^p_{\text{loc}}$. A sequence (f_n) is called *weakly (strongly) convergent in* $L^p_{\text{loc}}(\Omega)$ if (f_n) is weakly (strongly) convergent in $L^p(K)$, for every compact $K \subset \Omega$.

For $1 \leq p \leq \infty$, any $f \in L^p(\mathbb{R}^d)$,

$$\varphi_f(g) := \int_\Omega f(x)g(x)\,\mu(dx), \qquad \text{for all } g \in \mathcal{S}(\mathbb{R}^d)$$

defines a tempered distribution. With this identification of f and φ_f implicitly understood, the distributional derivatives of f are well-defined. If there is $g \in L^q(\Omega)$ such that

$$\partial^\alpha \varphi_f = \varphi_g$$

we say that the *weak (or distributional) derivative* of f is in $L^q(\Omega)$, and write $\partial^\alpha f \in L^q(\Omega)$. A similar meaning is given to $\partial^\alpha f \in L^q_{\text{loc}}(\Omega)$, for any multi-index α.

Definition 23.1 (Sobolev Spaces) *With the notations and conventions of above we define the Sobolev space $W^{1,p}(\Omega)$ (and its local variant $W^{1,p}_{\text{loc}}(\Omega)$) by*

$$W^{1,p}_{(\text{loc})}(\Omega) = \{f \in L^p_{(\text{loc})}(\Omega) \mid \partial^i f \in L^p_{(\text{loc})}(\Omega), i = 1, \ldots, d\}$$

$W^{1,p}(\Omega)$ is Banach space for the norm

$$\|f\|_{W^{1,p}(\Omega)} = \left[\|f\|^p_{L^p(\Omega)} + \sum_{i=1}^n \|\partial^i f\|^p_{L^p(\Omega)}\right]^{1/p}$$

Similarly, one defines $W^{n,p}(\Omega)$ and $W^{n,p}_{\rm loc}(\Omega)$ as the spaces of L^p functions with all partial derivatives of order $\le n$, in L^p. And the norm on $W^{n,p}(\Omega)$ is given by

$$\|f\|_{W^{n,p}(\Omega)} = \left[\sum_{0\le|\alpha|\le n} \|\partial^\alpha f\|^p_{L^p(\Omega)}\right]^{1/p}$$

A sequence (f_n) is *weakly (strongly) convergent in the Sobolev space* $W^{n,p}(\Omega)$ iff the sequences $(\partial^\alpha f_n)$ are weakly (strongly) convergent in $L^p(\Omega)$, for all $\alpha, 0 \le |\alpha| \le n$. And similarly for weak and strong convergence in the local Sobolev spaces.

Example:
Recall the definition of the Cantor set $C \subset [0,1]$. The complement of the Cantor set $D = [0,1] \setminus C$, is a countable union of closed intervals $I_{n,i}$, where $|I_{n,i}| = 3^{-n}$, and there are 2^n intervals of length 3^{-n}. The *Cantor function*, f_C, or *Devil's Staircase* vanishes on C, and is constant on the intervals $I_{n,i}$. The values on these intervals are such that on each interval $I_{n+1,i}$, f_C takes the average value of the adjacent intervals of the preceding level, i.e., $I_{n,j}$ and $I_{n,j+1}$ for some j, and $f_C(0) = 0$ and $f_C(1) = 1$.

Clearly the f'_C exists and vanishes on the interior of D, which is a set of Lebesgue measure 1. So, as an element of $L^p([0,1])$, $1 \le p \le \infty$, f_C is a.e.differentiable and its derivative is equivalent to the zero function, although f_C is far from being a constant function. However, this is *not* the weak derivative of f_C!

Exercise 23.1 *Let $1 \le p \le \infty$. Prove that if $f \in C^1(0,1])$, its weak derivative coincides with the usual pointwise definition of derivative.*

Exercise 23.2 *Let $1 \le p \le \infty$. Prove that if $f \in L^p([0,1], dx)$, and its weak derivative is identically zero, then there exists $c \in \mathbb{C}$, such that $f(x) = c$* a.e..

If $p = 2$ the Sobolev spaces are Hilbert spaces. To remind us of this fact $W^{n,p}(\Omega)$ is usually denoted by $H^n(\Omega)$.

Exercise 23.3 *Prove that $H^n(\Omega)$ is a Hilbert space for $n \ge 1$. I.e., define the appropriate inner product and prove that $H^n(\Omega)$ is complete (Hint: use that $L^2(\Omega)$ is a Hilbert space).*

Exercise 23.4 *Of the following functions in $L^2([-1,1], dx)$, which are in $H^1([-1,1], dx)$?*
i) $x \mapsto |x|$;
ii) $x \mapsto \mathrm{sign} x$;
iii) $x \mapsto \sum_{n=1}^{\infty} (1/n)^{3/2} \sin(2\pi n x)$.

Exercise 23.5 *For $\alpha = e_1, \ldots, e_d$, let ∂^α denote the linear operators defined by taking the weak derivatives of functions in the their domain $H^1(\Omega, dx) \subset L^2(\Omega, dx)$. Prove that the unbounded linear operators ∂^α are closed.*

23.2 Fourier characterization

Theorem 23.2 *Let $\hat{f}$ denote the Fourier transform of $f \in L^2(\mathbb{R}^d, dx)$. Then, $f \in H^1(\mathbb{R}^d, dx)$ iff $k \mapsto |k|\hat{f}(k)$ is in $L^2(\mathbb{R}^d)dx)$.*

23.3 The Sobolev Embedding Theorem

As an application of Fourier Series in Section 7.4.3 we proved a Sobolev Embedding Theorem for functions defined on the circle. Here, we will state a more general result. But first we need the notion of *regular domain*. In fact, there are many notions of regularity that one requires to be satisfied for the domain Ω, depending on the context. The differences among those are sometimes quite subtle. Any domain bounded by a smooth hypersurface (i.e., locally parametrically defined by smooth functions), will satisfy the necessary regularity for all the results mentioned in this chapter. Domains with outward pointing cups or needle-shaped protrusions are typical examples of domains with insufficient regularity and often the desired property does not hold for such domains. For more details on Sobolev Embedding Theorems and the regularity conditions they require, see, e.g., R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

It is customary to formulate the Sobolev Embedding Theorem in terms of a embedding $j : H^n(\Omega) \to C(\overline{\Omega})$ for regular, bounded open domains $\Omega \subset \mathbb{R}^d$, or $j : H^n(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$, in the case of $\Omega = \mathbb{R}^d$. Part of the theorem is that such an injective map can be defined if $n > d/2$, i.e., H^n functions are continuous if $n > d/2$. The theorem also says that the natural embedding j is a continuous linear transformation, i.e., mapping bounded sets into bounded sets, when each space is considered with its own norm.

Theorem 23.3 (Sobolev Embedding Theorem) *Suppose $n > d/2 + r$, $r \geq 0$. Then*
i) $H^n(\mathbb{R}^d) \subset C_0^r(\mathbb{R}^d)$;
ii) For every regular bounded open domain Ω

$$H^n(\Omega) \subset C^r(\overline{\Omega})$$

iii) For $\Omega = \mathbb{R}^d$, or Ω regular, bounded, and open, there exists a constant M, depending on d and n, such that

$$\|f\|_{\sup} \leq M\|f\|_{H^n}, \quad \text{for all } f \in H^n(\Omega)$$

23.4 Rellich's Theorem

Rellich's theorem is a common tool to prove that a set of functions in a Sobolev space is compact. We will state it without proof.

It is obvious from the definition that, if $m \geq n+1$, $H^m(\Omega) \subset H^n(\Omega)$. The corresponding embedding $j_{m,n}$ is again a bounded linear map. Rellich's Theorem states that $j_{m,n}$ is a *compact embedding* if $m \geq n+1$, i.e., $j_{m,n}$ maps bounded subsets of $H^m(\Omega)$ into pre-compact subsets of $H^n(\Omega)$, when each space is equipped with its own norm.

Theorem 23.4 (Rellich's Theorem) *Suppose $\Omega \subset \mathbb{R}^d$ is a regular bounded open domain, and suppose $m \geq n+1$. If A be a bounded subset of $H^m(\Omega)$, then A is pre-compact as a subset of $H^n(\Omega)$.*

Exercise 23.6 *Compare Rellich's Theorem with the Arzelá-Ascoli Theorem. (Hint: replace the condition of equicontinuity in the latter theorem by the stronger condition of a uniform bound on the first-order partial derivatives.)*