3 Topological Spaces

In the previous weeks we have generalized concepts from the finite-dimensional spaces $\mathbb{R}^n$ with the usual Euclidean metric, to general metric spaces. We are fundamentally interested in convergence of sequences and continuity of functions. These properties have been expressed in terms of the metric. And as an important auxiliary tool we explored the notion of compactness.

As a particularly useful example of infinite-dimensional metric spaces we studied the function spaces $C(K)$ in detail. We also mentioned that pointwise convergence of functions (e.g., real-valued functions defined on an interval) can generally not be expressed in terms of a metric on some function space, suggesting that there is an even more general framework for expressing convergence, continuity, and compactness. Indeed there is: topological spaces.

Here, we limit ourselves to the basic definitions. For more information on point set topology and its use, see, e.g.,
W. Rudin, *Functional Analysis*, 2nd ed, McGraw-Hill, New York 1991
or
J.L. Kelley and I. Namioka, *Linear Topological Spaces*, D. Van Nostrand Company, Princeton, NJ 1963

3.1 General definition of a topological space

Let X be an arbitrary set.

Definition 3.1 *A set $\mathcal{T}$ of subsets of X is a topology on X if*
i) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
ii) for every family $\{G_\alpha\} \subset \mathcal{T}$, $\bigcup_\alpha G_\alpha \in \mathcal{T}$, and
iii) for every finite family $\{G_1, \ldots, G_n\} \subset \mathcal{T}$, $\bigcap_{1\leq i\leq n} G_i \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called *open sets.* A set $A \subset X$ is called *closed* iff its complement in X, $A^c := X \setminus A$, is open. The pair $(X, \mathcal{T})$ is called a *topological space.*

Examples:

1. For any X, $\mathcal{T} = \{\emptyset, X\}$ is a topology and it is obviously the smallest $\mathcal{T}$ that satisfies Definition 3.1. $\mathcal{T}$ is called the *trivial topology* or *indiscrete topology.*

2. For any X, $\mathcal{T} = \mathcal{P}(X)$, the set of *all* subsets of X, obviously also satisfies Definition 3.1. It is called the *discrete topology.*

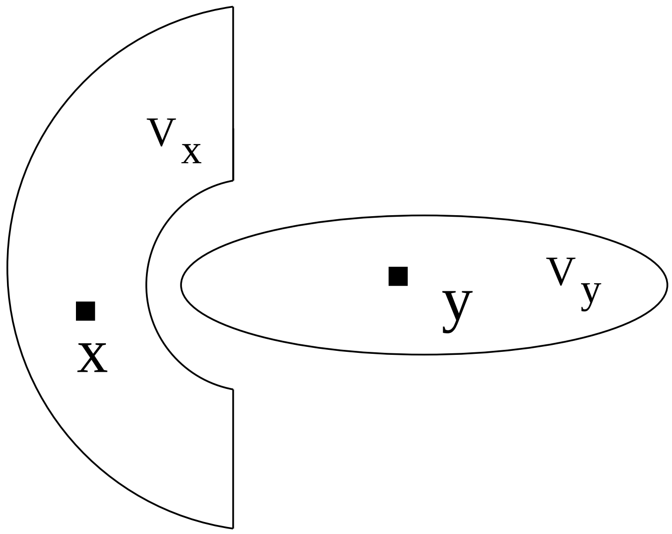

Figure 3.1: The Hausdorff property.

3. Let $(X, \mathcal{T})$ be a topological space and $X_0 \subset X$. Then $\mathcal{T}_0 = \{G_0 \subset X_0 \mid G_0 = G \cap X_0,$ for some $G \in \mathcal{T}\}$ is a topology on X_0. So, the open subsets of X_0 are all sets obtained by intersecting the open sets of X with X_0. This topology is called the *induced topology* or *relative topology* on X_0 and X_0 is said to be a topological subspace of X.

4. Let (X, d) be a metric space. Then the set of all open sets, as defined in Section 1, is a topology on X.

We will encounter many concrete examples of topologies later on.

Some more terminology:

- A set $V \subset X$ is called a *neighborhood* of a point $x \in X$ if there exixts an open set contained in V and containing x, i.e., if there exists $G \in \mathcal{T}$ such that $x \in G \subset V$.
- A topology $\mathcal{T}$ is called *Hausdorff* (or $(X, \mathcal{T})$ is called a Hausdorff space) if every pair of distinct points $x, y \in X$, has a pair of non-intersecting neighborhoods, i.e., there exists a neighborhood V_x of x and a neighborhood V_y of y such that $V_x \cap V_y = \emptyset$. Almost all topological spaces used in analysis are Hausdorff. E.g., all metric topologies are Hausdorff (why ?). On the other hand, if X has at least two elements, the trivial topology on X is not Hausdorff.

Next, we express the basic notions of convergence, continuity, and compactness in terms of open sets. Let X and Y be a topological spaces.

Definition 3.2 (convergence) *A sequence (x_n) in X converges to a limit $x \in X$ iff for every neighborhood V of x, there exists N such that for all $n \geq N$, $x_n \in V$.*

This definition says that eventually the sequence lies entirely in any neighborhood of x.

Exercise 3.1 *What are the convergent sequences in a topological space X with the trivial topology? And with the discrete topology?*

Definition 3.3 (continuity) *A function $f : X \to Y$ is continuous at $x \in X$ if for each neighborhood W of $f(x)$ there exists a neighborhood V of x such that $f(V) \subset W$. f is continuous on X if it is continuous at every $x \in X$.*

Theorem 3.4 *Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be two topological spaces and $f : X \to Y$. Then f is continuous iff for every $G \subset \mathcal{S}$, $f^{-1}(G) \in \mathcal{T}$.*

Exercise 3.2 *Prove Theorem 3.4*

If a function $f : X \to Y$ is one-to-one and onto (a bijection), and both f and f^{-1} are continuous, f is called a *homeomorphism*, and any two topological spaces X and Y such that there exists a homeomorphism $f : X \to Y$, are called *homeomorphic*. Homeomorphic spaces are indistinguishable as topological spaces. E.g., if X and Y are homeomorphic, one has that for any sequence (x_n) in X, (x_n) converges to x iff the sequence $(f(x_n))$ in Y converges to $f(x)$ (prove this).

Remark:
A bijection $f : X \to Y$ always has an inverse f^{-1}, but f^{-1} need not be continuous even if f is. E.g., consider $f : [0, 2\pi) \to S^1, f(\theta) = e^{i\theta}$, where $[0, 2\pi) \subset \mathbb{R}$ is considered with the topology induced by the standard topology on $\mathbb{R}$, and $S^1 \subset \mathbb{C}$ is the unit circle with the topology induced by the topology of the complex plane (or, equivalently, $\mathbb{R}^2$). f is continuous, but f^{-1} is not. For other examples see Section 3.3.

Definition 3.5 (compactness) *$K \subset X$ is compact if every open cover of K contains a finite subcover.*

Note that $K \subset X$ is compact in the topology $\mathcal{T}$ on X iff K is compact as a subset of itself for the induced topology of X on K.

Exercise 3.3 *Let (X, d_1) and (Y, d_2) be metric spaces. Prove that the definitions of convergence and continuity are equivalent to the ones previously given (Definition 1.2 and 2.1).*

The following proposition states that, in the case of metric spaces, compactness as defined here, is equivalent to (sequential) compactness as defined in Definition 1.9. Hence, we can keep using the term compactness (instead of sequential compactness) when we are considering metric spaces.

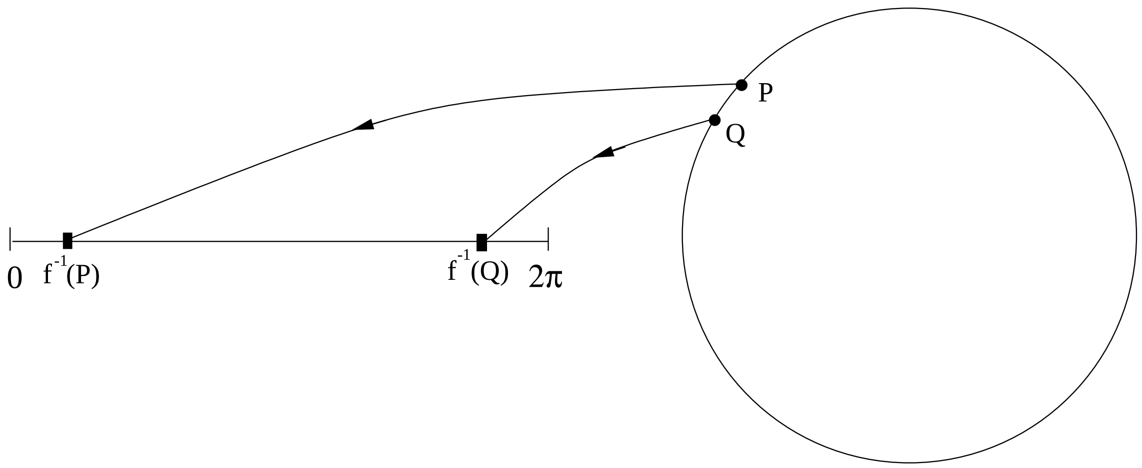

Figure 3.2: The interval and the circle are not homeomorphic. There are arbitrarily close points on the circle, P and Q in the figure, which have inverse images near the left and right end points of the interval respectively. Hence, the inverse images are not close and the inverse map cannot be continuous.

Proposition 3.6 *Let (K, d) be a metric space. Then K is compact in the sense of Definition 3.5 iff every sequence (x_n) in K has a convergent subsequence.*

Proof: We already proved in Theorem 2.16 that sequential compactness implies the finite-open-subcover property, i.e., compactness. It remains to prove the other direction, which we do by contraposition.

Suppose there is a sequence (x_n) in K without any convergent subsequence. Then the sequence necessarily contains an infinite number of distinct x_n. Therefore, without loss of generality, we may as well assume that all elements of (x_n) are distinct. We will show that the existence of such a sequence implies that K does not have the finite-open-subcover property, i.e., that K is not compact.

We start by constructing an open cover of K as follows. For each $x \in K$, either $x \in \{x_n \mid n \geq 1\}$ or $x \notin \{x_n \mid n \geq 1\}$. In both cases, as (x_n) has no convergent subsequences, only a finite number of the x_n belong to $B_{\epsilon_x}(x)$. Therefore, there is $\epsilon_x > 0$ such that $x_n \notin B_{\epsilon_x}(x)$ for all $x_n \neq x$. Clearly $\{B_{\epsilon_x}(x) \mid x \in K\}$ is an open cover of K. This cover has the property that every finite subcover $\{B_{\epsilon_x}(x) \mid x \in \{y_1, \ldots, y_k\} \subset K\}$ contains at most k elements of the sequence (x_n). As we assumed that all the elements of (x_n) are distinct, this implies that any finite subcover does not contain all elements of (x_n). Therefore, this open cover of K does not contain a finite subcover, which is in contradiction with the assumption that K is compact. ■

We repeat that in general, for non-metric topological spaces, sequential compactness does not imply compactness. In Proposition 1.8 we gave a characterization of closed sets, and the closure of a set, in terms of convergent sequences. Again, this breaks down for general topological spaces. Similarly, continuous functions $f : X \to Y$, in the case X and Y are metric spaces, were characterized in Proposition 2.3 as exactly those functions that preserve convergent sequences. Again, not so in arbitrary topological spaces. There are however objects that replace sequences, called nets, that can be used to express all the above-mentioned properties in an alternative way. We will not make use of nets in this course.

3.2 Bases of open sets

The set of all open sets can be huge and unwieldy. Recall that for metric spaces everything could be expressed in terms of open balls alone, a rather small subset of the open sets. In this section we introduce subsets of $\mathcal{T}$ (called *bases*) that can play the same role for an arbitrary topology as the open balls for a metric topology. We will also see that such special sets of open sets arise naturally when one defines new topologies.

Definition 3.7 *A family of open sets $\mathcal{B} \subset \mathcal{T}$ is a base for the topology $\mathcal{T}$, if for every $G \in \mathcal{T}$ there is a family of $B_\alpha \in \mathcal{B}$, such that $G = \bigcup_\alpha B_\alpha$.*

Examples:
1. The family of all open intervals $(a, b) \subset \mathbb{R}$, $a, b \in \mathbb{R}$, is a base for the standard topology on $\mathbb{R}$.
2. The family of all open intervals $(a, b) \subset \mathbb{R}$, with $a, b \in \mathbb{Q}$, is a base for the standard topology on $\mathbb{R}$.
3. Let (x, d) be a metric space. The set of all open balls $B_{1/n}(x)$, with $n \geq 1$ and $x \in A$, where A is dense in X, is a base for the metric topology on X.

Definition 3.8 *A family $\mathcal{N}$ of neighborhoods of a point $x \in X$ is called a neighborhood base (or a base of neighborhoods) of x if for any neighborhood V of x there exists a $W \in \mathcal{N}$ such that $W \subset V$.*

Example:
Let (X, d) be a metric space. The open (or closed) balls with center x and radius $1/n$, $n \geq 1$, form a neighborhood base for x.

Definition 3.9 (first countable) *A topological space X is first countable if every $x \in X$ has a countable neighborhood base.*

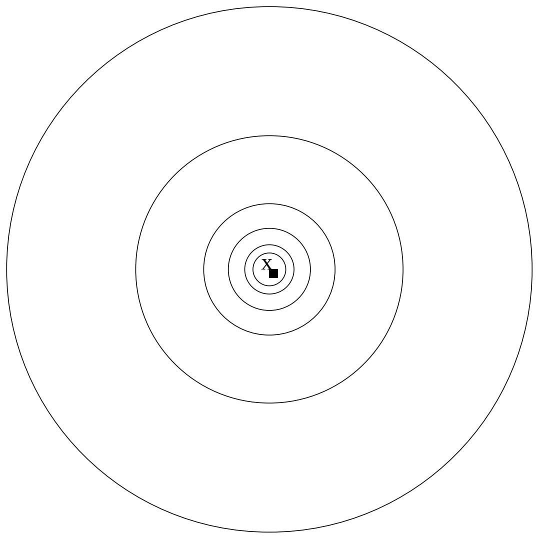

Figure 3.3: A base of neighborhoods of x.

Examples:
1. As the examples above show, any metric space is first countable.
2. Any X with the discrete topology is first countable.
3. Any X with the trivial topology is first countable.

Often, it is useful to *define* a topology in terms of a base or a neighborhood base for every point.

Examples:
1. Let X be the space of all real-valued functions on $[a, b] \subset \mathbb{R}$. Let $f \in X$, $\epsilon > 0$, $n \geq 1$, and $x := \{x_1, \ldots, x_n\} \subset [a, b]$. Define a subset $B_{f,x,\epsilon} \subset X$ by

$$B_{f,x,\epsilon} = \{g \in X \mid |g(x_i) - f(x_i)| < \epsilon, \text{ for all } i = 1, \ldots, n\}$$

Then, define $\mathcal{T}$ as the smallest set of subsets of X satisfying the conditions of Definition 3.1 and containing $B_{f,x,\epsilon}$, for all $f \in X$, $\epsilon > 0$, and finite sets $x \subset X$. This defines the topology of pointwise convergence of functions in the sense that $f_n \to f$ in the topological space $(X, \mathcal{T})$ iff $f_n(x) \to f(x)$ for all $x \in [a, b]$. The sets $B_{f,x,\epsilon}$ are a neighborhood base for $f \in X$. A base for the topology can be obtained by considering arbitrary finite intersections of sets of the form $B_{f,x,\epsilon}$. In fact, it is easy to see that it is sufficient to consider sets of the form

$$\{g \in X \mid |g(x_i) - f(x_i)| < \epsilon_i, \text{ for all } i = 1, \ldots, n\}$$

for all $f \in X$, $\epsilon_i > 0$, and finite sets $x \subset X$.
2. Let V be a linear space. A function $p : V \to \mathbb{R}$ is called a *seminorm* on V if it

has the following properties:
i) $p(x) \geq 0$ for all $x \in V$.
ii) $p(ax) = |a|p(x)$, for all $x \in V$, $a \in \mathbb{R}$.
iii) $p(x+y) \leq p(x) + p(y)$, for all $x, y \in V$.
If, in addition, p satisfies $p(x) > 0$ for all $x \neq 0$, p is called a *norm*. We already know how to make a normed linear space into a metric space (and hence a topological space). More generally, suppose P is a set of seminorms on a linear space V. One can then define a neighborhood base for any $x \in V$ parametrized by $\epsilon > 0$ and finite subsets $\{p_1, \ldots, p_n\} \subset P$:

$$B_{\epsilon, p_1, \ldots, p_n}(x) = \{y \in V \mid p_1(x-y) < \epsilon, \ldots, p_n(x-y) < \epsilon\}$$

We will define various interesting topologies on linear spaces exactly in this way, i.e., by defining appropriate sets of seminorms. Note that the topology of pointwise convergence on the space of all real-valued functions on some common domain is of this type. If the set of seminorms used is uncountable, chances are that the topology is not first countable and non-metric.

Theorem 3.10 *A family of open sets $\mathcal{B} \subset \mathcal{T}$ is a base for the topology $\mathcal{T}$ iff for every $x \in X$, $\mathcal{B}$ contains a base of neighborhoods for x.*

Proof: Suppose $\mathcal{B}$ is a base for $\mathcal{T}$. Then, if V is a neighborhood of x then there exists $G \in \mathcal{T}$, such that $x \in G \subset V$. As $\mathcal{B}$ is a base, there are B_α such that $\bigcup_\alpha B_\alpha = G$. Therefore, there exists $B \in \mathcal{B}$, such that $x \in B$ and $B \subset V$. We conclude that $\mathcal{B}$ contains a base of neighborhoods of x.

Conversely, if a family of open sets $\mathcal{B}$ contains a base of neighborhoods for every $x \in X$, then, for every open set $G \in \mathcal{T}$, and any $x \in G$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subset G$. Therefore, $\bigcup_x B_x = G$, and we have proved that $\mathcal{B}$ is a base for the topology. ■

3.3 Comparing topologies

It is clear that different metrics on a space X can lead to one and the same notion of convergence of sequences and continuity of functions. E.g., if d_1 an d_2 are two metrics on X such that for all $x, y \in X$, $d_1(x,y) = 2d_2(x,y)$, this will certainly be the case. And in general, if two metrics lead to exactly the same set of open sets all topological properties should be the same. Ee will see that indeed the notion of topology (as the set of open sets) is more fundamental than, e.g., a norm or a metric.

As noted in the previous section, a topology if often given indirectly by a base of neighborhoods for every point. A natural question is then: how can one compare topologies by investigating a base of neighborhoods for them? When are the topologies the same?

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on the same space X. Then, $\mathcal{T}_2$ is called *finer* than $\mathcal{T}_1$ if $\mathcal{T}_1 \subset \mathcal{T}_2$, i.e., if it has more open sets (one then also says that $\mathcal{T}_1$ is *coarser* than $\mathcal{T}_2$). Of course, not every pair of topologies are comparable, i.e., it is quite possible that $\mathcal{T}_1 \not\subset \mathcal{T}_2$ and $\mathcal{T}_2 \not\subset \mathcal{T}_1$.

Proposition 3.11 *Let X and Y be two spaces, each with two topologies, $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{S}_1, \mathcal{S}_2$ respectively. If $f : (X, \mathcal{T}_1) \to (Y, \mathcal{S}_1)$ is continuous and*
i) if $\mathcal{T}_2$ is finer than $\mathcal{T}_1$, then $f : (X, \mathcal{T}_2) \to (Y, \mathcal{S}_1)$ is continuous
ii) if $\mathcal{S}_2$ is coarser than $\mathcal{S}_1$, then $f : (X, \mathcal{T}_1) \to (Y, \mathcal{S}_2)$ is continuous

Proof: These statements are a direct consequence of the general definition of continuity (Definition 3.3), or, more directly, of Theorem 3.4. ■

The most regular bijection from a space X to itself one can imagine is the identity map $\mathrm{id} : X \to X, x \mapsto \mathrm{id}(x) = x$. As a map $\mathrm{id}^{-1} = \mathrm{id}$, and for every topology $\mathcal{T}$ on X, $\mathrm{id}^{-1} = \mathrm{id} : (X, \mathcal{T}) \to (X, \mathcal{T})$ are continuous, and hence *id* is a homeomorphism. The situation changes drastically when we consider two different topologies on X, say $\mathcal{T}_1$ and $\mathcal{T}_2$. E.g., consider

$$\mathrm{id} : (X, \{\emptyset, X\}) \to (X, \mathcal{P}(X))$$

Then, if X contains at least two points, id is not continuous at any point $x \in X$. More generally, as the following theorem shows, the continuity of the map id is a perfect tool for comparing topologies.

Theorem 3.12 *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Then the map* $\mathrm{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ *is continuous iff $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.*

Proof: This is a direct consequence of Theorem 3.4. ■

Corollary 3.13 $\mathrm{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ *is a homeomorphism iff $\mathcal{T}_1 = \mathcal{T}_2$.*

Theorem 3.14 *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Then $\mathcal{T}_1 = \mathcal{T}_2$ iff the following is true: for all topological spaces $(Y, \mathcal{S})$, a function $f : (X, \mathcal{T}_1) \to (Y, \mathcal{S})$ is continuous iff $f : (X, \mathcal{T}_2) \to (Y, \mathcal{S})$ is continuous.*

Proof: If $\mathcal{T}_1 = \mathcal{T}_2$, then the statement about continuous functions $f : X \to Y$ is trivial. For the other direction we use Corollary 3.13. So, we have to show that $\mathrm{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is a homeomorphism. This follows by once taking $(Y, \mathcal{S}) = (X, \mathcal{T}_1)$, and once $(Y, \mathcal{S}) = (X, \mathcal{T}_2)$. ■

The next result is of practical importance, because often topologies are defined indirectly through bases of neighborhoods for every point. So, it is particularly useful to be able to distinguish (or identify) topologies on the basis of bases of neighborhoods alone.

Theorem 3.15 *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. And suppose that for every $x \in X$ there exist bases of neighborhoods of x, $\mathcal{N}_1$ and $\mathcal{N}_2$, for $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively, such that for every $V \in \mathcal{N}_1$ there is a $W \in \mathcal{N}_2$ such that $W \subset V$. Then $\mathcal{T}_2$ is finer than $\mathcal{T}_1$.*

Proof: We simply use Theorem 3.12. It is obvious that under the condition stated $\mathrm{id} : (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ is continuous and the result follows. ∎

Corollary 3.16 *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Then, $\mathcal{T}_1 = \mathcal{T}_2$ iff for every $x \in X$ there are bases of neighborhoods of x, $\mathcal{N}_1$ and $\mathcal{N}_2$, for $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively, such that for every $V \in \mathcal{N}_1$ there is a $W \in \mathcal{N}_2$, such that $W \subset V$, and, vice versa, there are bases of neighborhoods $\mathcal{M}_1$ and $\mathcal{M}_2$, for $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively, such that for every $W \in \mathcal{M}_2$ there is a $V \in \mathcal{M}_2$, such that $V \subset W$.*

For metric spaces one can charatcerize the topology also by the set of all convergent sequences. Note that in general, the same topology can be obtained from many different metrics. Metrics leading to the same topology, i.e., the same set of open sets, are called *equivalent* .

Theorem 3.17 *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two metric topologies on X, defined by metrics d_1 and d_2 respectively. Then, $\mathcal{T}_1 = \mathcal{T}_2$, i.e., d_1 and d_2 are equivalent metrics, iff $x_n \to x_0$ in $(X, d_1) \Leftrightarrow x_n \to x_0$ in (X, d_2).*

Proof: The proof is a direct application of Corollary 3.13. ∎

For general topological spaces there is the analogous theorem with sequences replaced by nets.

3.4 Comments on non-metric topologies

1. The notion of Cauchy sequence is defined in terms of a metric. Therefore, also completeness is defined only for metric spaces.
2. Recall that the closure $\overline{A}$ of a set $A \in X$ is defined as the smallest closed set containing A. For metric spaces $\overline{A}$ can be obtained from A through addition of all the limits of converging sequences with elements in A (See Section 1.4). For a non-metric topological space this procedure might fail. In fact, the set thus obtained, sometimes called the *sequential closure* of A and denoted by $\overline{A}^S$, need not be closed. In general one has $A \subset \overline{A}^S \subset \overline{A}$. An example of $\overline{A}^S \neq \overline{A}$ is the following. Define

$$f_{m,n}(x) = [\cos(m!\pi x)]^{2n}$$

on $[0,1]$, for all $m,n \geq 1$. The following pointwise limits are easy to compute:

$$\lim_{n\to\infty} f_{m,n}(x) := f_m(x) = \begin{cases} 1 & \text{if} \quad x = k/m!, k = 0,\ldots,m! \\ 0 & \text{else} \end{cases}$$

and

$$\lim_{m\to\infty} f_n(x) := f(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{else} \end{cases} \tag{3.1}$$

For $A = \{f_{m,n} \mid m,n \geq 1\}$, we then have $f_n \in \overline{A}^S$, and certainly also

$$f \in \overline{\overline{A}^S}^S$$

The claim is that $f \notin \overline{A}^S$, and hence $\overline{A}^S \neq \overline{\overline{A}^S}^S$ and therefore, $\overline{A}^S \neq \overline{A}$. To show that $f \notin \overline{A}^S$ one can resort to the following proposition:

Proposition 3.18 *Any function $f : [0,1] \to \mathbb{R}$, that is the pointwise limit of continuous functions, is continuous on a dense subset of $[0,1]$.*

As f defined in (3.1) is nowhere continous in $[0,1]$, it is not the pointwise limit of some sequence of functions $f_{m,n}$, which are all continuous. This example also proves that the topology of pointwise convergence on the real-valued functions on $[0,1]$ is not metric.