4 Completeness and the Contraction Mapping Theorem

Recall that a metric space (X, d) is called *complete* iff every Cauchy sequence in X converges. Our next objective is the *Contraction Mapping Theorem* that we will first state and discuss and prove later in this section.

References for this section are
G.F. Simmons, *Introduction to Topology and Modern Analysis*, McGraw-Hill, New York, 1963,

and

J.E. Marsden and M. J. Hoffman, *Elementary Classical Analysis*, 2nd ed., W.H. Freeman and Company, New York, 1993.

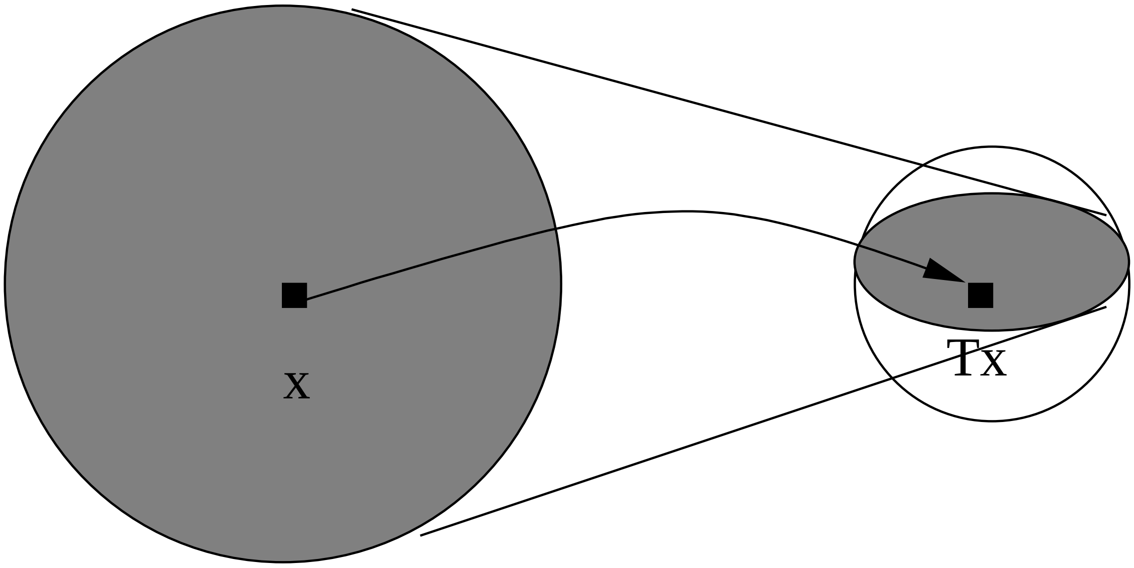

Figure 4.1: T is a contraction.

Definition 4.1 (contraction) *Let (X, d) be a metric space. A map $T : X \to X$ is a contraction mapping (or contraction), if there exists a constant c, $0 \leq c < 1$, such that*

$$d(T(x), T(y)) \leq cd(x, y) \tag{4.1}$$

for all $x, y \in X$.

Exercise 4.1 *Prove that if T is a contraction on a metric space X, then T is uniformly continuous.*

Keep in mind that in some texts a map satisfying (4.1) with $c = 1$ is also called a contraction. Sometimes a map satisfying (4.1) for some $c < 1$ is then called *strictly contracting* or a a *strict contraction.*

Theorem 4.2 (Contraction Mapping Theorem) *If $T : X \to X$ is a contraction mapping and (X, d) is complete, then the equation*

$$T(x) = x \tag{4.2}$$

has exactly one solution $x \in X$.

A solution of (4.2) is called a *fixed point* of T. So, strict contractions have exactly one fixed point. The Contraction Mapping Theorem is only one example of what are more generally called fixed-point theorems. There are fixed-point theorems for maps satisfying (4.1) with $c = 1$, or even for arbitrary continuous transformations on certain metric spaces. In general, however, $c < 1$ is needed for uniqueness or even existence of a fixed point. E.g., consider the metric space $X = \{0, 1\}$, with the metric determined by $d(0, 1) = 1$. The map T defined by $T(0) = 1, T(1) = 0$, satisfies (4.1) with $c = 1$. But T does not have any fixed points. On the other extreme, the identity map on any metric space trivially satisfies (4.1) with $c = 1$, and every point is a fixed point.

It is worth noting that (4.2), and hence its solutions, do not refer to the metric d. If one finds a metric on X such that T is a contraction according to Definition 4.1, one can apply the theorem and obtain existence and uniqueness of the fixed point *if* X is complete for that same metric d. And there is the catch: the frustrating situation may occur that for none of the metrics for which one manages to prove that T is a contraction, X is complete. (X, d) might be complete for some metric, but for that metric T is then not a contraction. This can be an indication that the natural setting for the fixed point problem at hand is not X, but another, larger, space $(\tilde{X}, \tilde{d})$, called the *completion* of (X, d). Therefore, before we prove Theorem 4.2, we will discuss the notion of *completion* of a metric space.

4.1 Completion of metric spaces

A metric space $(\tilde{X}, \tilde{d})$ is called the *completion* of (X, d) (*the* completion, rather than *a* completion in anticipation of Theorem 4.3), if the following conditions are satisfied:

1. $(\tilde{X}, \tilde{d})$ is a complete metric space.

2. There is an embedding $i : X \to \tilde{X}$, satisfying

$$\tilde{d}(i(x), i(y)) = d(x, y) \tag{4.3}$$

for all $x, y \in X$.

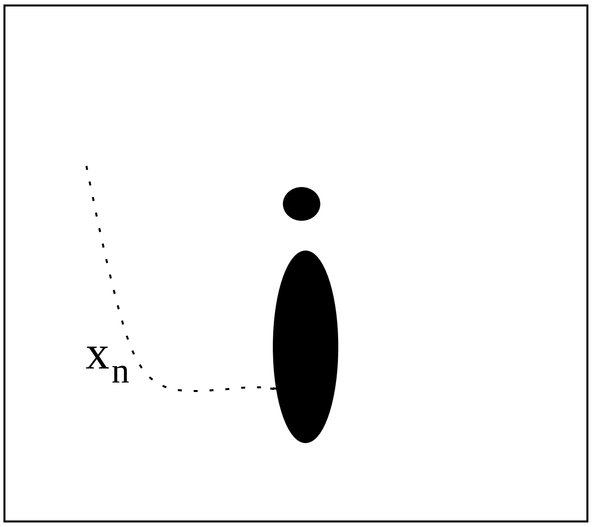

Figure 4.2: The sequence x_n seems to converge, but the candidate limit point is not an element of the space, i.e., x_n converges to a "hole" in the space.

3. $i(X)$ is dense in $\tilde{X}$, i.e., $\overline{i(X)} = \tilde{X}$.

A map i satisfying (4.3) is called *isometric*, or an *isometry*. Two metric spaces (X, d_X) and (Y, d_Y) are *isomorphic* if there exists an isometric bijection ($\equiv$ *isomorphism*) between them. They are then indistinguishable as metric spaces. Note that an isomorphism of metric spaces is also a homeomorphism of the corresponding topological spaces.

Note that isometries are automatically continuous and one-to-one. Some authors include the condition that the map be onto in the definition of isometry. With the latter convention isometries are isomorphisms of metric spaces. Do not let this confuse you. It is usually clear from the context whether an isometry is assumed to be onto or not.

We now have the terminology to state the theorem about the completion of metric spaces.

Theorem 4.3 *Every metric space (X, d) has a completion $(\tilde{X}, \tilde{d})$. The completion is unique up to isomorphism, i.e., any two completions $(\tilde{X}, \tilde{d})$ and $(\hat{X}, \hat{d})$ of (X, d) are isomorphic as metric spaces.*

Proof: We first construct a completion $(\tilde{X}, \tilde{d})$ of (X, d). Consider the set C of all Cauchy sequences in X, and define an equivalence relation on C by

$$x \sim y \quad \text{iff} \lim_{n\to\infty} d(x_n, y_n) = 0$$

for any two Cauchy sequences $x = (x_n), y = (y_n) \in C$. Note that if two Cauchy sequences x and y are convergent, then their limits are the same iff $x \sim y$. Verify that $\sim$ is indeed an equivalence relation. Let $\tilde{X}$ be the set of equivalence classes in C, i.e., $\tilde{X} = C/\sim$. Next, we define $\tilde{d} : \tilde{X} \times \tilde{X} \to \mathbb{R}$ by

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} d(x_n, y_n) \tag{4.4}$$

where (x_n) and (y_n) are two representatives of $\tilde{x}$ and $\tilde{y}$ respectively. It is elementary to check that $(d(x_n, y_n))_{n\geq 1}$ is a Cauchy sequence of real numbers, and hence the limit exists. For this definition to make sense it is also crucial that the limit is independent of the chosen representatives. This can be checked as follows: suppose $(x_n), (x'_n) \in \tilde{x}$, and $(y_n), (y'_n) \in \tilde{y}$. Then, by the triangle inequality,

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x'_n) + d(x'_n, y'_n) + d(y'_n, y_n) \\ d(x_n, y_n) &\geq d(x'_n, y'_n) - d(x_n, x'_n) - d(y'_n, y_n) \end{aligned}$$

Taking the limit $n \to \infty$ of these inequalities, and using that $(x_n) \sim (x'_n)$, and $(y_n) \sim y'_n)$, we see that the limit in (4.4) is indeed independent of the representatives. So, $\tilde{d}$ is well-defined, and it is straightforward to check that it defines a metric on $\tilde{X}$.

Next, we check that the metric space $(\tilde{X}, \tilde{d})$ is indeed a completion of (X, d).
1) To check that $(\tilde{X}, \tilde{d})$ is complete, consider an arbitrary Cauchy sequence $(\tilde{x}_n)$ in $\tilde{X}$. If necessary, we can pick a subsequence, denoted again by $(\tilde{x}_n)$, such that

$$\tilde{d}(\tilde{x}_n, \tilde{x}_m) \leq \frac{1}{n}, \qquad \text{for all } m > n. \tag{4.5}$$

For each element of this sequence, choose a representative $(x_{n,k}) \in \tilde{x}_n$ such that,

$$d(x_{n,k}, x_{n,l}) < \frac{1}{n} \qquad \text{for all } k, l \geq n \tag{4.6}$$

This is possible because any representative is a Cauchy sequences and by selecting an appropriate subsequence, which is automatically also Cauchy and in the same equivalence class, we can make sure that (4.6) holds. We claim that the sequence $(x_{k,k})$ is a Cauchy sequence and that it is a representative of the limit of $(\tilde{x}_n)$. Indeed, for all $k, l \geq n$, note that

$$\begin{aligned} d(x_{n,k}, x_{k,k}) &\leq d(x_{n,k}, x_{n,l}) + d(x_{n,l}, x_{k,l}) + d(x_{k,l}, x_{k,k}) \\ &\leq \frac{2}{n} + d(x_{n,l}, x_{k,l}) \end{aligned}$$

where we have used (4.6). By the definition of $\tilde{d}$, and (4.5), we have

$$\lim_{l\to\infty} d(x_{n,l}, x_{k,l}) = \tilde{d}(\tilde{x}_n, \tilde{x}_k) \leq \frac{1}{n}$$

By combining these estimates we obtain that $(\tilde{x}_n)$ converges to the equivalence class of $(x_{k,k})_{k=1}^{\infty}$:

$$\begin{aligned}\lim_{n\to\infty} \tilde{d}((\tilde{x}_n), \widetilde{(x_{k,k})}) &= \lim_{n\to\infty}\lim_{k\to\infty} d(x_{n,k}, x_{k,k}) \\ &\leq \limsup_{n\to\infty} \frac{3}{n} = 0\end{aligned}$$

This shows that $(\tilde{X}, \tilde{d})$ is complete.
2) An isometric embedding $i : X \to \tilde{X}$ can be defined by mapping $x \in X$ to the equivalence class of Cauchy sequences containing the constant sequence $x_n = x$. That this map is isometric follows from

$$\tilde{d}(i(x), i(y)) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x, y) = d(x, y)$$

3) $i(X)$ is dense in $\tilde{X}$ is seen as follows. Let $\tilde{x} \in \tilde{X}$ with representative (x_n). Because (x_n) is a Cauchy sequence the sequence of constant sequences $(y_{n,k} = x_n)_{k=1}^{\infty}$ satisfies

$$\lim_{n\to\infty} \tilde{d}(\tilde{(y_n)}, \tilde{(x_n)}) = 0$$

To conclude the proof of the theorem we still have to show that the completion is unique up to isomorphism, which we do in the following way. Let $(\tilde{X}, \tilde{d})$ be the completion of (X, d) constructed above and let $(\hat{X}, \hat{d})$ be any other completion, with its own isometric embedding $j : X \to \hat{X}$. We will extend j to a map $\tilde{j} : \tilde{X} \to \hat{X}$, in the sense that $\tilde{j} \circ i = j$. To define $\tilde{j}$ let us pick a representative (x_n) for any $\tilde{x} \in \tilde{X}$ and define

$$\tilde{j}(\tilde{x}) = \lim_{n\to\infty} j(x_n)$$

Because j is isometric $(j(x_n))$ is a Cauchy sequence in $\hat{X}$, and therefore converges. For the same reason the limit will be independent of the representative. It is also easy to see that j is isometric:

$$\hat{d}(\tilde{j}(\tilde{x}, \tilde{j}(\tilde{y})) = \lim_n \hat{d}(j(\tilde{x}_n), j(\tilde{y}_n)) = \lim_n d(x_n, y_n) = \tilde{d}(\tilde{x}, \tilde{y})$$

By chosing the constant sequences as representatives of $\tilde{x}, x \in X$ we see that, for $x \in X$, $\tilde{j}(i(x)) = j(x)$. To check that $\tilde{j}$ is onto, note that $\hat{X}$ contains all Cauchy sequences in X. So, we have $\hat{X} = \overline{j(X)} \subset \tilde{j}(\tilde{X}) \subset \hat{X}$, which shows that $\tilde{j}$ is onto. ∎

It is slightly annoying that $\tilde{X}$ is a space of sequences, rather than some more direct extension of X. E.g., there is no guarantee that the completion of a space of functions will still be a space of functions, just containing more functions than the original one. We will come back to this issue in concrete examples.

Examples:

1. $\mathbb{R}$ is the completion of $\mathbb{Q}$ with the usual metric. This is one of the standard ways of introducing (constructing, and mathematically defining) the real numbers. Note that when you write a real number in decimal notation, you in fact are giving a Cauchy sequence of rational numbers converging to it.

2. Consider the following metric on $C([a, b])$:

$$d(f, g) = \left[\int_a^b (f(x) - g(x))^2 \, dx \right]^{1/2}$$

This is a very useful metric but $C([a, b])$, with this metric, is not complete. Its completion is denoted by $L^2([a, b])$, the space of all real-valued square-integrable functions on $[a, b]$. The dirty secret is that it is really not a space of functions, but a space of equivalence classes of (measurable) functions. Many of the usual operations with functions can still be defined, independent of the representative, but the evaluation of an element $f \in L^2([a, b])$ in a point, $f(x_0)$ *cannot* be unambiguously defined! We will meet L^2 again, and eventually come to think of it as a very nice space, consisting of objects as good as functions.

4.2 The Contraction Mapping Theorem

Recall the statement of Theorem 4.2: *If $T : X \to X$ is a contraction mapping and (X, d) is complete, then the equation $T(x) = x$ has exactly one solution $x \in X$.*
Proof: The proof is contructive in the sense that we will explicitly construct a sequence converging to the fixed point.

Let x_0 be any point in X and defined a sequence (x_n) by $x_1 = Tx_0$, $x_{n+1} = Tx_n$ (to alleviate the notations we often ommit the parentheses around the argument of a map). Denote by T^n the n^{th} power (n^{th} iterate) of T. So, $x_n = T^n x_0$.

First, we show that (x_n) is a Cauchy sequence. Let $n \geq m \geq 1$.

$$\begin{aligned}
d(x_n, x_m) &= d(T^n x_0, T^m x_0) \\
&\leq c^m d(T^{n-m} x_0, x_0) \\
&\leq c^m \left[d(T^{n-m} x_0, T^{n-m-1} x_0) + d(T^{n-m-1} x_0, T^{n-m-2} x_0) + \cdots + d(Tx_0, x_0) \right] \\
&\leq c^m \left[\sum_{k=0}^{n-m-1} c^k \right] d(x_1, x_0) \\
&\leq c^m \left[\sum_{k=0}^{\infty} c^k \right] d(x_1, x_0) \\
&\leq d(x_1, x_0) \frac{c^m}{1 - c}
\end{aligned}$$

from which it follows that (x_n) is Cauchy. As X is complete x_n converges to a limit $x \in X$. That x is a fixed point of T follows using the continuity of T:

$$Tx = T\lim_n x_n = \lim_n Tx_n = \lim_n x_{n+1} = x$$

The uniqueness of the fixed point is also a consequence of the contraction property. Suppose x and y are two fixed points. Then

$$0 \leq d(x, y) = d(Tx, Ty) \leq cd(x, y)$$

As $c < 1$, this implies $d(x, y) = 0$, and hence $x = y$. So, the fixed point is unique. ■