6 Banach Spaces

In this section we focus on a very important class of complete metric spaces called Banach spaces. A *Banach Space* is linear space V, equipped with a norm, usually denoted by $\|\cdot\|$, that is a complete metric space for the metric derived from this norm ($d(x, y) = \|x - y\|$).

If you do not remember much about linear spaces, see e.g.,
P.D. Lax, *Linear Algebra*, John Wiley & Sons, New York 1997
or
P.R. Halmos, *Finite-dimensional Vector Spaces*, 2nd ed D. Van Nostrand Company, Princeton 1958.

For more information on Banach spaces see
W. Rudin, *Functional Analysis*, 2nd ed, McGraw-Hill Inc., New York 1991
or
G.F. Simmons, *Introduction to Topology and Modern Analysis*, McGraw-Hill, New York, 1963.

Recall that a *norm* on a linear space V over the field of scalars $\mathbb{R}$ (or $\mathbb{C}$) is a function $\|\cdot\| : V \to \mathbb{R}$ with the following properties:
(N1) $\|x\| \geq 0$, for all $x \in V$. (positivity)
(N2) $\|\lambda x\| = |\lambda|\|x\|$, for all $x \in V$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$). (homogeneity)
(N3) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in V$. (triangle inequality)
(N4) $\|x\| = 0 \Rightarrow x = 0$. (strict positivity)
If the (N1–3) hold but not necessarily (N4), $\|\cdot\|$ is called a *seminorm*.

All linear spaces that we will consider will be either over $\mathbb{R}$ or $\mathbb{C}$, respectively called *real* and *complex* linear spaces.

In a normed linear space it makes sense to consider series. The series

$$\sum_{n=1}^{\infty} x_n$$

of elements $x_n \in X$ is said to *converge* to $x \in X$ if the sequence of partial sums (S_n)

$$S_N = \sum_{n=1}^{N} x_n$$

converges to x. The series is said to *converge absolutely* iff the series

$$\sum_{n=1}^{\infty} \|x_n\|$$

converges as a series in $\mathbb{R}$, i.e., $\sum_{n=1}^{\infty} \|x_n\| < +\infty$.

Proposition 6.1 *Let X be a normed linear space. Then, X is complete iff every absolutely convergent series converges.*

Exercise 6.1 *Prove Proposition 6.1.*

6.1 Bounded linear maps

Let X and Y be two normed linear spaces (not necessarily complete). As usual we will denote both norms by $\|\cdot\|$.

Definition 6.2 *A linear map $T : X \to Y$ is bounded if there exists a constant $M \geq 0$ such that*

$$\|Tx\| \leq M\|x\|, \quad \text{for all } x \in X \tag{6.1}$$

Theorem 6.3 *A linear map $T : X \to Y$ is bounded iff it is continuous.*

Proof:
1) Bounded $\Rightarrow$ Continuous. For all $x, y \in X$ we have

$$\|Tx - Ty\| = \|T(x - y)\| \leq M\|x - y\|$$

where M is a constant for which (6.1) holds. Therefore, one can take $\delta = \epsilon/M$ in the definition of continuity.
2) Continuous $\Rightarrow$ Bounded. If T is continuous, then T is continuous at $0 \in X$. So, for all $\epsilon > 0$, there exists $\delta > 0$ such that $\|x\| \leq \delta \Rightarrow \|Tx - T0\| = \|Tx\| \leq \epsilon$. Pick any $\delta > 0$. Then, for any $0 \neq x \in X$, $\tilde{x}$ defined by

$$\tilde{x} = \frac{\delta}{\|x\|}x$$

satisfies $\|\tilde{x}\| \leq \delta$. Therefore, $\|T\tilde{x}\| \leq \epsilon$. It follows that for all $x \in X$

$$\|Tx\| = \|\frac{\|x\|}{\delta}T\tilde{x}\| \leq \frac{\|x\|}{\delta}\epsilon$$

Hence, T satisfies Definition 6.2 with $M = \epsilon/\delta$. ∎

Note that we have also proved that if T is continuous at $0 \in X$, then T is continuous on the whole of X. This can also be seen as a direct consequence of linearity.

The set of all bounded linear maps $T : X \to Y$ is denoted by $\mathcal{B}(X, Y)$.

Next, we define the *null space* or *kernel* of a linear map $T \in \mathcal{L}(X, Y)$, denoted by $\ker T$, and the *range* of T, denoted by $\text{ran}T$.

$$\begin{aligned} \ker T &= \{x \in X \mid Tx = 0\} \\ \text{ran}T &= \{y \in Y \mid \text{ there exists } x \in X, \text{ such that } Tx = y\} \end{aligned}$$

Theorem 6.4 *For any $T \in \mathcal{B}(X,Y)$,* $\ker T$ *is a closed linear subspace of X.*

Proof:
• $\ker T$ *is a linear subspace:* For all $\lambda \in \mathbb{R}$ or $\mathbb{C}$, and $x, y \in \ker T$, one has

$$T(x + \lambda y) = Tx + \lambda T y = 0$$

and therefore, $x + \lambda y \in \ker T$ (here, we only used the fact that T is linear, not that it is bounded).
• $\ker T$ *is closed:* As X is a metric space, we only have to show that for any sequence of elements $x_n \in \ker T$, converging to x, we necesarily have $x \in \ker T$. This follows from the continuity of T:

$$Tx = T \lim_n x_n = \lim_n T x_n = \lim_n 0 = 0$$

■

When $X = Y$ one usually simplifies the notation: $\mathcal{L}(X,X) = \mathcal{L}(X)$, and $\mathcal{B}(X,X) = \mathcal{B}(X)$. In that case the linear maps T are referred to as *linear transformations* of X, or *linear operators* on X.

Another important special case is when $Y = \mathbb{R}$ or $\mathbb{C}$ (depending on whether X is a real or complex linear space). The elements of $\mathcal{L}(X, \mathbb{C})$ and $\mathcal{B}(X, \mathbb{C})$ are then called linear *functionals* (or linear forms), and bounded ($\equiv$ continuous) linear functionals on X, respectively. $\mathcal{L}(X, \mathbb{C})$ is also called the *algebraic dual*, and $\mathcal{B}(X, \mathbb{C})$ the *topological dual* space of X. Somewhat confusingly, both are commonly denoted by X^*.

Examples:

1. Consider $X = C([a,b])$. For any $\rho \in L_1([a,b])$, the following formula defines a continuous linear functional φ on X:

$$\varphi(f) = \int_a^b f(x)\rho(x)\,dx \tag{6.2}$$

2. Not all continuous linear functionals are of the form (6.2). E.g., for any $x_0 \in [a,b]$, evaluation of f at x_0 is a continuous linear functional. I.e., define

$$\delta_{x_0}(f) = f(x_0),$$

then δ_{x_0} is a continuous linear functional on $C([a,b])$.

3. An example of a discontinuous linear functional on $C([0,1])$ is the following. It is easy to see that the set of monomials $M = \{x^n \mid n \geq 0\}$ is a linearly independent set of vectors in $C([0,1])$. By a general theorem (see, e.g., the reference by Simmons given at the beginning of this section), $C([0,1])$ has a Hamel basis , H, containing M. A *Hamel basis*, is a maximal linearly independent set of vectors. Any f, can be uniquely written as finite linear combination of vectors in H, i.e., for any $f \in C([0,1])$, there is a unique set of elements, $h_i \in H$, $i = 1, \dots, n$, and unique non-zero coeffcients c_i such that

$$f = c_1 h_1 + \cdots c_n h_n \tag{6.3}$$

For each $n \geq 0$ define $a_n(f)$ by

$$a_n(f) = \begin{cases} c_i & \text{if } h_i = x^n \\ 0 & \text{else} \end{cases}$$

Due to the uniqueness of the decompositon this is well-defined. Define a linear functional φ on $C([0,1])$ by

$$\varphi(f) = \sum_{n=1}^{\infty} n a_n(f)$$

As, for each f, only a finite number of terms in this sum are non-zero, this is well-defined. However, it is quite obvious that φ is unbounded. E.g.,

$$\sup\{|\varphi(f)| \mid \|f\| = 1\} \geq \sup\{|\varphi(f)| \mid f \in M\} = \sup\{n \geq 1\} = +\infty$$

Hence, by Theorem 6.3, this implies that φ is discontinuous. In fact, with this example we have shown that every infinite-dimensional linear space has discontinuous linear functionals defined on it. On the other hand, as will be shown in Theorem 6.14, all linear functionals on a finite-dimensional linear space are continuous.

6.2 Topologies on $\mathcal{B}(X,Y)$

Let X and Y be normed linear spaces. Then, for any $T \in \mathcal{B}(X,Y)$ the following defines a norm on $\mathcal{B}(X,Y)$:

$$\|T\| = \inf\{M \mid \|Tx\| \leq M\|x\|, \text{for all } x \in X\} \tag{6.4}$$

Check that this indeed defines a norm. An alternative expression for $\|T\|$ is

$$\|T\| = \sup\{\|Tx\| \mid \|x\| \leq 1\} \tag{6.5}$$

When $X \neq \{0\}$ one also has

$$\|T\| = \sup\{\|Tx\| \mid \|x\| = 1\} \tag{6.6}$$

$$\|T\| = \sup\{\frac{\|Tx\|}{\|x\|} \mid x \neq 0\} \tag{6.7}$$

This norm on $\mathcal{B}(X,Y)$ is called the *uniform* or *operator* norm, or sometimes just the *usual* norm. And if for a sequence of $T_n \in \mathcal{B}(X,Y)$ one has $\lim_n \|T_n - T\| = 0$, for some $T \in \mathcal{B}(X,Y)$, one says that T_n converges uniformly to T, or that it converges in the uniform (operator) topology.

It is clear that $\mathcal{L}(X)$ is not only a linear space, but that it is also an algebra with the product given by the composition of linear maps. The following theorem implies that $\mathcal{B}(X)$ is a subalgebra of $\mathcal{L}(X)$, i.e., the product of two bounded linear maps is bounded.

Theorem 6.5 *Let X, Y, and Z be normed linear spaces. For any $T \in \mathcal{B}(X,Y)$ and $S \in \mathcal{B}(Y,Z)$, one has that $ST \in \mathcal{B}(X,Z)$, and*

$$\|ST\| \leq \|S\|\,\|T\| \tag{6.8}$$

Proof: For all $x \in X$ we have

$$\|STx\| \leq \|S\|\,\|Tx\| \leq \|S\|\,\|T\|\,\|x\|$$

■

Note that (6.8) also shows that the product of bounded linear maps is a continuous map $\mathcal{B}(Y,Z) \times \mathcal{B}(X,Y) \to \mathcal{B}(X,Z)$.

Theorem 6.6 *If Y is a Banach space, then $\mathcal{B}(X,Y)$ is a Banach space for the uniform norm.*

Proof: We have to prove that every Cauchy sequence in $\mathcal{B}(X,Y)$ converges (to a limit in $\mathcal{B}(X,Y)$). Let (T_n) be a Cauchy sequence. We have that for all $x \in X$

$$\|T_n x - T_m x\| \leq \|T_n - T_m\|\,\|x\|$$

which shows that the sequence $(T_n x)$ is a Cauchy sequence in Y. As Y is complete, there exists $y \in Y$ such that $T_n x \to y$. It is then straightforward to check that $Tx = y$ defines a linear map $X \to Y$. Next, we show that T is bounded. For any $\epsilon > 0$, let N_ϵ be such that $\|T_n - T_m\| < \epsilon/2$ for all $n, m \geq N_\epsilon$. Take $n \geq N_\epsilon$. Then, for all $x \in X$, $\|x\| = 1$, there exist $m(x) \geq N_\epsilon$ such that

$$\|T_n x - Tx\| \leq \|T_n x - T_{m(x)} x\| + \|T_{m(x)} x - Tx\| \leq \epsilon/2\|x\| + \epsilon/2 \leq \epsilon \tag{6.9}$$

Therefore, for all $x \in X$, $\|x\| = 1$, $\|Tx\| \leq \|T_n x\| + \epsilon \leq \|T_n\| + \epsilon$, and T is bounded. Finally, from (6.9) it also follows that $\lim_n \|T_n - T\| = 0$, Hence, $T_n \to T$ in the uniform norm. ■

Exercise 6.2 *Prove that if $T_n \to T$ uniformly, then $\|T_n\| \to \|T\|$.*

The following theorem says that a bounded linear map has a unique bounded extension to the closure (or completion) of its domain. Theorem is very useful when one defines linear maps (one only has to define them on a dense subspace) or when one has to show that two linear maps are equal (one has to show equality only on a dense subspace).

Theorem 6.7 *Let X be a normed linear space and let Y be a Banach space. For any dense linear subspace $X_0 \subset X$ and any $T_0 \in \mathcal{B}(X_0, Y)$, there exists a unique $T \in \mathcal{B}(X, Y)$ such that $Tx_0 = T_0 x_0$, for all $x_0 \in X_0$.*

Proof: We first define a map $T \in \mathcal{B}(X, Y)$ as follows. For all $x \in X$, there is a sequence (x_n) in X_0 converging to x. Define

$$Tx = \lim_n T_0 x_n$$

This limit exists because T_0 is bounded and Y is complete. Also, any two sequences (x_n) and (x'_n) converging to x, satisfy

$$\lim_n \|x_n - x'_n\| \leq \lim_n \|x_n - x\| + \lim_n \|x - x'_n\|$$

and therefore

$$\|\lim_n T_0 x_n - \lim_n T_0 x'_n\| = \lim_n \|T_0 x_n - T_0 x'_n\| \leq \lim_n \|T_0\| \, \|x_n - x'_n\| = 0$$

and Tx is independent of the choice of the sequence (x_n) converging to x.

T is an extension of T_0, i.e., $Tx_0 = T_0 x_0$, for all $x_0 \in X_0$, because we can take the constant sequence $x_n = x_0$, for all n, to define Tx_0. The linearity of T is a consequence of the linearity of the T_n, the fact that X_0 is a linear subspace, and the independence of the chosen sequence that we just proved.

Next, we prove that T is bounded. It follows from

$$\|Tx\| = \lim_n \|T_0 x_n\| \leq \lim_n \|T_0\| \, \|x_n\| = \|T_0\| \, \|x\|$$

that $\|T\| \leq \|T_0\|$. In fact, as $Tx_0 = T_0 x_0$, for $x_0 \in X_0$, we must have $\|T\| = \|T_0\|$.

Finally, we show that T is the unique bounded linear map $X \to Y$, coinciding with T_0 on X_0. Suppose S is another such map. Then, for all $x \in X$, and sequences $x_n \to x$, $x_n \in X_0$,

$$Sx = \lim_n Sx_n = \lim_n T_0 x_n = Tx$$

For the first inequality we used that S is bounded by assumption, for the second that S is an extension of T_0, and the third is the definition of T. ■

The following exercise asks you to think about the importance of the linearity of T_0 in the previous theorem. Is there a property that can replace it for more general functions?

Exercise 6.3 *What can one say about continuous extensions of functions $f : X_0 \to Y$, where X_0 is a dense subset of a metric space X and Y is a complete metric space?*

Next, we look at an example of a convergent sequence of bounded linear maps on $C([0,1])$, and also an example of a divergent sequence, which, however, convergence in a different topology on $\mathcal{B}(C([0,1]))$.

Example:
Let $X = C([0,1])$, equipped with the supremum norm. For $n = 1, 2, \ldots$, let $k_n(x,y)$ be a sequence of real-valued continuous functions on $[0,1] \times [0,1]$ such that

$$\lim_n \left\{ \sup_{x\in[0,1]} \int_0^1 |k_n(x,y)|\, dy \right\} = 0 \tag{6.10}$$

For $f \in X$, define $K_n \in \mathcal{B}(X)$ by

$$K_n f(x) = \int_0^1 k_n(x,y) f(y)\, dy \tag{6.11}$$

Then $K_n \to 0$ in the uniform norm. Indeed, for all $f \in X$, we have

$$\begin{aligned} |K_n f(x)| &= \left| \int_0^1 k_n(x,y) f(y)\, dy \right| \\ &\le \int_0^1 |k_n(x,y)|\, |f(y)|\, dy \\ &\le \|f\| \int_0^1 |k_n(x,y)|\, dy \\ &\le \|f\| \sup_x \int_0^1 |k_n(x,y)|\, dy \end{aligned}$$

which, by (6.10) implies that

$$\|K_n\| \le \sup_{x\in[0,1]} \int_0^1 |k_n(x,y)|\, dy \to 0$$

An example of functions k_n satisfying (6.10) is

$$k_n(x,y) = xy^n$$

Next, Consider another sequence of linear transformations $K_n \in \mathcal{B}(X)$, of the form (6.11), but with the functions k_n given by

$$k_n(x,y) = \sin(n\pi y)$$

It is not hard to prove (see the Riemann-Lebesgue Lemma below), that, for any $f \in C([0,1])$

$$\lim_n \int_0^1 \sin(n\pi y)\, f(y)\, dy = 0$$

And, therefore,

$$\lim_n \|K_n f\| = 0 \tag{6.12}$$

for all $f \in X$. On the other hand, for $f_n = \sin(n\pi x)$,

$$\|f_n\| = 1, \quad \text{but} \quad \|K_n f_n\| = 1/2$$

which implies that $\|K_n\| \geq 1/2$, and so K_n does *not* converge to 0 in the uniform norm. This motivates us for the definition of a second notion of convergence of linear maps (and a second topology on $\mathcal{B}(X,Y)$, distinct from the uniform norm topology whenever X is infinite-dimensional).

Definition 6.8 (Strong Convergence) *A sequence (T_n) in $\mathcal{B}(X,Y)$ converges strongly (or in the strong operator topology) to $T \in \mathcal{B}(X,Y)$, if for all $x \in X$*

$$\lim_n T_n x = Tx$$

Note that, by (6.12), K_n of the previous example, converges strongly to zero.

Theorem 6.9 *If $T_n \to T$ uniformly, then $T_n \to T$ strongly.*

Exercise 6.4 *Prove Theorem 6.9.*

Appendix: The Riemann-Lebesgue Lemma

In this appendix we consider complex-valued functions on a finite interval $[a,b]$. Recall the definition of the L_1 norm: for all $f \in C([a,b])$

$$\|f\|_1 = \int_a^b |f(x)|\, dx$$

The completion of $C([a,b])$ with this norm is denoted by $L_1([a,b])$, and, loosely speaking, it is the space of all functions f for which $\|f\|_1$ is finite, and its elements are called L_1 functions. It is easy to see that the product of an L_1 function and a bounded continuous function is again L_1. The functions considered can be either real- or complex-valued, in which case $L_1([a,b])$ is a real or complex Banach space respectively.

The Riemann-Lebesgue Lemma can be considered as a statement about the Fourier coefficients of L_1 functions.

Theorem 6.10 (Riemann-Lebesgue Lemma) *For any $f \in L_1([0, 2\pi])$ one has*

$$\lim_{n\to\pm\infty} \int_0^{2\pi} f(x)e^{inx}\, dx = 0$$

Proof: It is sufficient to prove this for f in a dense subspace of $L_1([0, 2\pi])$ (e.g., use Theorem 6.7).

A convenient choice for the dense subspace is the space of all trigonometric polynomials of the form

$$P_n(x) = \sum_{k=-n}^{n} a_k e^{ikx}$$

the a_k are arbitrary complex coefficients. By Theorem 2.13 the trigonometric polynomials are dense in $C(T)$ with the uniform norm (T is the unit circle). As the L^1-norm is weaker than the uniform norm, the trigonometric polynomials are also dense in $C(T)$ with the L^1 norm. As $C(T)$ is dense in $L_1([0, 2\pi])$ by construction, the trigonometric polynomials are also dense in we conclude $L_1([0, 2\pi])$. For any such P_n it is clear that

$$\int_0^{2\pi} P_n(x)e^{-ikx}\, dx = 2\pi a_k$$

And, therefore,

$$\int_0^{2\pi} P_n(x)e^{-ikx}\, dx = 0$$

for $|k| > n$, and so

$$\lim_{k\to\pm\infty} \int_0^{2\pi} P_n(x)e^{-ikx}\, dx = 0$$

■

6.3 Remarks on Finite-Dimensional Banach Spaces

Let X be a finite-dimensional linear space over $\mathbb{R}$ or $\mathbb{C}$. *Finite-dimensional* means that there is a finite set, $\{e_1, \ldots, e_n\} \subset X$, with the property that for any $x \in X$, there are constants $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ or $\mathbb{C}$, such that

$$x = \alpha_1 e_1 + \cdots + \alpha_n e_n \tag{6.13}$$

In fact, there is n, such that one can choose n *linearly independent* vectors $e_1, \ldots, e_n$ such that every $x \in X$ can be written as a linear combination of them. n is then the *dimension* of X. Linear independence of $e_1, \ldots, e_n$ is equivalent to the property that the coefficients α_i in (6.13) are uniquely determined by x.

Our main objective in these remarks on finite-dimensional Banach spaces is to show that every finite-dimensional normed linear space is a Banach space (i.e., is complete), and that, whenever X is finite-dimensional, every $T \in \mathcal{L}(X,Y)$ is bounded, i.e., $\mathcal{L}(X,Y) = \mathcal{B}(X,Y)$.

For the rest of this chapter, let X be a finite-dimensional normed linear space over $\mathbb{R}$ or $\mathbb{C}$. For simplicity of notation we will use $\mathbb{C}$ as the number field in the statements and proofs below, with the understanding that everywhere one can replace $\mathbb{C}$ by $\mathbb{R}$ and obtain the corresponding result about real linear spaces.

Lemma 6.11 *Let $\{e_1, \ldots, e_n\}$ be a basis for X. For any $i, 1 \leq 1 \leq n$, define a linear functional $\varphi_i : X \to \mathbb{C}$ by*

$$\varphi_i(x) = \alpha_i$$

where α_i is the coefficient of e_i appearing in the decomposition of x (6.13). Then, φ_i is bounded.

Proof: We will show that there exists a constant $m > 0$ such that

$$m \sum_{i=1}^{n} |\alpha_i| \leq \|x\| \tag{6.14}$$

It then follows that, for any i,

$$|\varphi_i(x)| = |\alpha_i| \leq \sum_{i=1}^{n} |\alpha_i| \leq \frac{1}{m}\|x\|$$

and, as $m > 0$, φ_i is bounded.

Clearly, it is sufficient to prove (6.14) for $x \in X$ such that $\sum_{i=1}^{n} |\alpha_i| = 1$ (use linearity and uniqueness of the coefficients α_i). The set

$$C = \{(\alpha_i) \in \mathbb{C}^n \mid \sum_{i=1}^{n} |\alpha_i| = 1\}$$

is a closed and bounded, and hence compact subset of $\mathbb{C}^n$ (Heine-Borel, Theorem 1.10). Define a function $f : C \to X$ by

$$f((\alpha_i)) = \sum_{i=1}^{n} \alpha_i e_i = x$$

As

$$\|f((\alpha_i) - f((\beta_i))\| = \left\| \sum_{i=1}^{n} (\alpha_i - \beta_i) e_i \right\| \leq \sum_{i=1}^{n} |\alpha_i - \beta_i| \|e_i\|$$

f is continuous, hence $(\alpha_i) \mapsto \|f((\alpha_i))\|$ is continuous, and, by Theorem 2.5, $\|f\|$ attains its infimum. Define

$$m = \min_{(\alpha_i)\in C} |f((\alpha_i))|$$

and suppose $|f((\alpha_i^*))| = m$. It follows from the linear independence of the basis vectors e_i that $m \neq 0$, and hence $m > 0$. For $(\alpha_i) \in C$ we have

$$m \sum_{i=1}^{n} |\alpha_i| = m \leq \|x\|$$

which implies (6.14). ■

Theorem 6.12 *Every finite-dimensional normed linear space is a Banach space.*

Proof: We need to show that every Cauchy sequence $(x_n)_{n\geq 1}$ in a finite-dimensional normed linear space X converges. Let $\{e_1, \dots, e_k\}$ be a basis for X. Consider the sequences $(\alpha_{i,n})_{n\geq 1}$ in $\mathbb{C}$, where

$$x_n = \sum_{i=1}^{k} \alpha_{i,n} e_i$$

Then, for $i = 1, \dots k$, $(\alpha_{i,n})_{n\geq 1}$ is Cauchy. Indeed

$$|\alpha_{i,n} - \alpha_{i,m}| = |\varphi_i(x_n - x_m)| \leq M\|x_n - x_m\|$$

where φ_i is the linear function considered in the previous lemma, and M is a common bound for the φ_i. It follows that there are $\beta_i \in \mathbb{C}, i = 1 \dots, k$, such that

$$\lim_n \alpha_{i,n} = \beta_i$$

Define $y = \sum_{i=1}^{k} \beta_i e_i$. Then

$$\|x_n - y\| \leq \sum_{i=1}^{k} |\alpha_{i,n} - \beta_i| \|e_i\|$$

and therefore, $x_n \to y$. ■

Corollary 6.13 *Every finite-dimensional linear subspace of a Banach space is closed.*

Proof: Use the previous theorem and the following Exercise. ■

Exercise 6.5 *Let (X, d) be a complete metric space, and $X_0 \subset X$. Prove that (X_0, d) is complete iff X_0 is a closed subset of X.*

A linear map $T : \mathbb{C}^n \to \mathbb{C}^m$, can be defined by specifying a matrix $T = (t_{ij})$, such that

$$T(e_i) = \sum_{j=1}^{m} t_{ji} f_j, \quad \text{for all } i = 1, \ldots, n$$

where $\{e_i\}$ and $\{f_j\}$ are bases for $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. It is then easy to see that, with the usual norms on $\mathbb{C}^n$ and $\mathbb{C}^m$, T is bounded. E.g., $\|T\| \leq \max_{ij} |t_{ji}|$. The following theorem is a generalization of this observation.

Theorem 6.14 *Let X and Y be normed linear spaces. If X is finite-dimensional, $\mathcal{L}(X, Y) = \mathcal{B}(X, Y)$.*

Proof: Let $\{e_1, \ldots, e_n\}$ be a basis for X, and $x = \sum_{i=1}^{n} \alpha_i e_i \in X$. Then

$$\|Tx\| \leq \sum_{i=1}^{n} |\alpha_i| \, \|Te_i\| \leq M\|x\| \sum_{i=1}^{n} \|Te_i\|$$

where M is a common bound for the maps φ_i of Lemma 6.11. It follows that

$$\|T\| \leq M \sum_{i=1}^{n} \|Te_i\|$$

■

The next theorem shows that, although one can define many different norms on finite-dimensional linear spaces such as $\mathbb{R}^n$ and $\mathbb{C}^n$, there is only one topology derived from a norm on these spaces.

Theorem 6.15 *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X. If X is finite-dimensional, then there exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1, \quad \textit{for all } x \in X.$$

Proof: Consider the identity map $\mathbb{1} \in \mathcal{L}((X, \|\cdot\|_1), (X, \|\cdot\|_2))$. By Theorem 6.14 $\mathbb{1}$ is bounded. So, there exists a constant M such that

$$\|x\|_2 \leq M\|x\|_1, \quad \text{for all } x \in X.$$

Therefore, one can take $C_2 = M$. By the same argument with the roles of 1 and 2 interchanged one finds C_1. ■

Definition 6.16 *Let X be a linear space. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are called equivalent iff there exists constants $C_1, C_2 > 0$ such that*

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1, \tag{6.15}$$

for all $x \in X$.

With this definition Theorem 6.15 can be rephrased as follows: any two norms on a finite-dimensional linear space are equivalent. In the following theorem we prove that equivalent norms, and only equivalent norms, generate the same topology on X. In combination with Theorem 6.15 this then proves that there is only one norm topology on a finite-dimensional linear space.

Theorem 6.17 *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a (not necessarily finite-dimensional) linear space X. Then, $\|\cdot\|_1$ and $\|\cdot\|_2$ generate the same topology on X iff $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms.*

Proof: Consider the identity map $\mathrm{id} : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2))$. The topologies generated by the two norms are the same iff id and id^{-1} are continuous (Corollary 3.13). As id is linear, id is continuous iff it is bounded. By the same reasoning as in the proof of the previous theorem, the boundedness of the identity map is equivalent to the existence of constants C_1 and C_2 such that (6.15) holds. ■

Corollary 6.18 *A finite-dimensional linear space has exactly one norm topology.*

Exercise 6.6 *Prove that*
a) *in general, different norms on X lead to different but equivalent norms on $\mathcal{B}(X)$;*
b) *the set $\mathcal{B}(X)$ itself is independent of the chosen norm.*