8 Special orthonormal bases 1. Fourier series

What makes Hilbert space so powerful in applications is the possibility to express a problem in a suitable o.n. basis, e.g., one of the bases of orthogonal polynomials introduced in Section 7.3.3. But orthogonal polynomials are by no means the only interesting o.n. bases. One of the most widely used is the *Fourier basis*, consisting of trigonometric functions, which is the subject of this chapter. Here, we will introduce the Fourier basis, which is the mathematical formulation underlying Fourier series, and investigate its general properties. Later, in Chapter 9, we will explore a variety of applications of Fourier series. In Chapter 10 we will give a brief introduction to yet another class of important special o.n. bases, the so-called *wavelets*.

8.1 Completeness of the Fourier basis

Our main objective is to prove that $\{e_n \mid n \in \mathbb{Z}\}$ with

$$e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx} \tag{8.1}$$

is an o.n. basis of $L^2([0, 2\pi])$. Of course one can then construct an o.n. basis for $L^2([a, b])$, for any $a < b \in \mathbb{R}$, by a simple affine transformation of the functions (i.e., by a shift and a rescaling of the argument).

In concreto, this then means that for any $f \in L^2([0, 2\pi])$, we have

$$f(x) = \sum_{n\in\mathbb{Z}} a_n \frac{1}{\sqrt{2\pi}} e^{inx} = \frac{a_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} (a_n \frac{a_0}{\sqrt{2\pi}} e^{inx} + a_{-n} \frac{a_0}{\sqrt{2\pi}} e^{-inx})$$

where the equality means convergence of the partial sums to f in the L^2-norm.

Our proof consists of the following steps:

1. $\{e_n \mid n \in \mathbb{Z}\}$ is an orthonormal set:

$$\begin{aligned} \int_0^{2\pi} \overline{\frac{1}{\sqrt{2\pi}} e^{imx}} \frac{1}{\sqrt{2\pi}} e^{inx}\, dx &= \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)x}\, dx \\ &= \begin{cases} 1 & \text{if } n = m \\ \frac{e^{i(n-m)2\pi} - e^{i(n-m)0}}{i(n-m)} = 0 & \text{if } m \neq n \end{cases} \end{aligned}$$

2. Let P denote the set of all finite linear combinations of functions e_n. These are the *trigonometric polynomials*. According to Theorem 7.11 one of the equivalenty conditions for completeness is that P is dense in $L^2([0, 2\pi])$, i.e., $\overline{P} = [\{e_n\}] = L^2([0, 2\pi])$. By construction, the continuous functions $C([0, 2\pi])$ are dense in $L^2([0, 2\pi])$. Therefore, it is sufficient to show that P is dense in the continous functions with the L^2-norm.

3. We will, in fact, prove a stronger result, namely that P is dense in the *periodic* continuous functions (i.e, continuous f such that $f(0) = f(2\pi)$, or, equivalently, the continuous functions defined on the unit circle, which we denote by $C(T)$), with the *uniform* norm, which is a useful fact in its own right. Compare Theorem 8.1 with the Weierstrass approximation theorem (Theorem 1.11). Both theorems are a special case of a much more general density thereom, called the Stone-Weirestrass Theorem (see W. Rudin, Functional Analysis).

Exercise 8.1 *a) Prove that P is dense in the periodic continuous functions on $[0, 2\pi]$ with the uniform norm, implies P is dense in the space of all continous functions on $[0, 2\pi]$ with the L^2-norm.*
b) Is P dense in the space of all continous functions $[0, 2\pi]$ with the uniform norm?

Theorem 8.1 *The trigonometric polynomials are dense in $(C(T), \|\cdot\|_{\sup})$.*

Proof: We parametrize T by the interval $[0, 2\pi]$ with 0 and 2π identified. So, $C(T)$ is then the space of continuous function $[0, 2\pi] \to \mathbb{C}$, that satisfy $f(0) = f(2\pi)$.

For this proof we introduce the notion of *approximate identity*. We will define a sequence of operators $I_n : C(T) \to P$ such that

$$\lim_{n\to\infty} \|I_n f - f\|_{\sup} = 0, \quad \text{for all } f \tag{8.2}$$

It is clear that such a sequence of operators approximates the identity operator in the uniform norm on the continuous functions. Obviously, if we can find such a sequence of operators the theorem follows.

We claim that the following I_n satisfy (8.2).

$$I_n f(x) = \int_0^{2\pi} \varphi_n(x-y) f(y)\, dy$$

where

$$\varphi_n(x) = c_n(1 + \cos x)^n$$

where the c_n are positive constants chosen such that

$$\int_0^{2\pi} \varphi_n(x)\, dx = 1, \quad \text{for all } n \tag{8.3}$$

Note that $\varphi_n \in P$ for all n:

$$\varphi_n(x) = \sum_{k=-n}^{n} a_n e^{inx}$$

First of all, the I_n are well-defined maps $C(T) \to P$. Indeed, for any $f \in C(T)$ we have

$$\begin{aligned} I_n f(x) &= \int_0^{2\pi} \sum_{k=-n}^{n} a_k e^{ik(x-y)} f(y)\,dy \\ &= \sum_{k=-n}^{n} a_k e^{ikx} \int_0^{2\pi} e^{-iky} f(y)\,dy \\ &= \sum_{k=-n}^{n} b_k e^{ikx} \end{aligned}$$

where

$$b_k = a_k \int_0^{2\pi} e^{-iky} f(y)\,dy$$

Next, we have to check the uniform convergence (8.2). This is similar to the proof of Weierstrass' approximation theorem. We use that any continous f on a compact (here T), is bounded and uniformly continuous. We also use (8.3), and note that

$$\int_0^{2\pi} \varphi_n(x-y) f(y)\,dy = \int_0^{2\pi} \varphi_n(y) f(x-y)\,dy$$

Pick $\epsilon > 0$ such that $|x-y| < \epsilon \Rightarrow |f(x) - f(y)| \leq \delta$. Then

$$\begin{aligned} |I_n f(x) - f(x)| &= \left| \int_0^{2\pi} \varphi_n(y)(f(x-y) - f(x))\,dy \right| \\ &= \left| \int_{|y|\leq\epsilon} \varphi_n(y)(f(x-y) - f(x))\,dy \right| \end{aligned}$$

$$+ \left| \int_{|y|>\epsilon} \varphi_n(y)(f(x-y) - f(x))\,dy \right|$$

$$\leq \delta \int_{|y|\leq\epsilon} \varphi_n(y) + 2\|f\|_{\sup} \int_{|y|>\epsilon} \varphi_n(y)\,dy$$

By taking the sup over x, and by using

$$\lim_n \int_{|y|>\epsilon} \varphi_n(y)\,dy = 0, \qquad \text{for all } \epsilon > 0 \tag{8.4}$$

we obtain (8.2). ∎

Exercise 8.2 *Prove (8.4).*

We have proved that for $f \in L^2([0, 2\pi])$

$$\lim_{N\to\infty} \| \frac{1}{2\pi} \sum_{n=-N}^{N} \left[\int_0^{2\pi} e^{-iny} f(y)\, dy \right] e^{inx} - f(x) \|_{L^2} = 0 \tag{8.5}$$

Theorem 7.6 implies that (8.5) is the best possible approximation with a trigonometric polynomial, in the sense of L^2. It is interesting to ask for other kinds of convergence of the partial sums of the series to the function. E.g., it can be proved that they also converge pointwise a.e. to $f(x)$. The proof of this is beyond the scope of this course.

One can also show that for *smooth* f the convergence of the partial sums is *uniform* (see Section 8.2 for a more detailed result. Another interesting question is how the partial sums behave in the neighborhood of a point of discontinuity of f. See H. Dym, and H.P. McKean, *Fourier Series and Integrals*, Section 1.6 for a discussion of this question (the Gibbs Phenomenon).

It is sometimes useful to modify the o.n. basis $\{e_n(x)\}$ a little bit. It is clear that the functions

$$\begin{aligned} e_0(x) &= \frac{1}{\sqrt{2\pi}} \\ \frac{1}{\sqrt{2}}(e_n + e_{-n}) &= \frac{1}{\sqrt{\pi}} \cos(nx), \quad n \geq 1 \\ \frac{1}{i\sqrt{2}}(e_n - e_{-n}) &= \frac{1}{\sqrt{\pi}} \sin(nx), \quad n \geq 1 \end{aligned}$$

together, also form an o.n. basis. This is useful, e.g., because, in this basis, a real-valued function will have real Fourier coeffients. Another useful property of the o.n. basis with cos and sin functions is the following. Consider all functions to be defined on $[-\pi, \pi]$. A function f is called *even* if $f(-x) = f(x)$, for all x, and f is called *odd* if $f(-x) = -f(x)$, for all x. It is then easy to verify that even functions f have an expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx)$$

while for odd f

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx) \quad .$$

The Fourier coefficients of f are often denoted by $\hat{f}(n)$.

Exercise 8.3 *Prove that the sets* $\{e_n \mid n \geq 1\}$ *defined by*

$$e_n(x) = \sqrt{\frac{2}{\pi}} \sin nx, \quad 0 \leq x \leq \pi$$

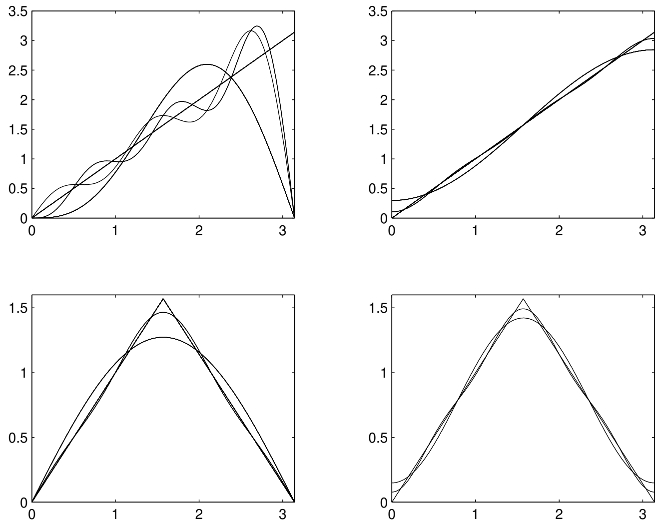

Figure 8.1: Sine and Cosine series for two piecewise linear functions. Note that the difference in the qualtity of the approximations.

and $\{f_n \mid n \geq 0\}$ *defined by*

$$f_0(x) = \sqrt{\frac{1}{\pi}}, \quad f_n(x) = \sqrt{\frac{2}{\pi}} \cos nx, \quad 0 \leq x \leq \pi, n \geq 1,$$

are two o.n. bases of $L^2([0,\pi])$.

8.2 Fourier series of $f \in C^p(T)$

Our aim is to show that if f has p continuous derivatives, with $p = 1, 2, 3, \ldots$, then the partial sums S_N of its Fourier series converge uniformly to f, and to give an estimate of the speed of convergence. We will use the following observation. If $f \in C^1(T)$, we can relate the Fourier coefficients of f' to the ones of f by integration by parts:

$$\begin{aligned} \widehat{f'}(n) &:= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} f'(x)\, dx \\ &= \frac{1}{\sqrt{2\pi}}[f(2\pi) - f(0)] - \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} (-in) e^{-inx} f(x)\, dx \\ &=: in\hat{f}(n) \end{aligned} \tag{8.6}$$

By induction it follows from from (8.6) that if $f \in C^p(T)$ then $\widehat{f^{(p)}}(n) = (in)^p \hat{f}(n)$.

Note that for relation (8.6) to hold it is, in fact, sufficient that $f' \in L^2(T)$. We will return to this later when we discuss weak derivatives and distributions.

Theorem 8.2 *Let $p \geq 1$. If $f \in C^p(T)$ the partial sums*

$$S_N(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$$

satisfy

$$\|S_N - f\|_{\sup} \leq \frac{\|f^{(p)}\|_{L^2}}{\sqrt{2\pi p} N^{p-1/2}}$$

In particular, the partial sums converge uniformly to f.

Proof: Note that $f \in C(T)$ and $S_N \in C(T)$ for all N. Due to the completeness of $C(T)$ for the supremum norm it is, therefore, sufficient to prove that for all $M \geq N$

$$\|S_N - S_M\|_{\sup} \leq \frac{1}{\sqrt{2\pi p}} \frac{1}{N^{p-1/2}} \|f^{(p)}\|_{L^2}$$

For this we use (8.6) and the inequalities of Cauchy-Schwarz and Bessel:

$$\begin{aligned}
\|S_N - S_M\|_{\sup} &\leq \frac{1}{\sqrt{2\pi}} \sum_{N<|n|\leq M} |\hat{f}(n)| \\
&= \frac{1}{\sqrt{2\pi}} \sum_{N<|n|\leq M} |\widehat{f^{(p)}}(n)| \frac{1}{|n|^p} \\
&\leq \frac{1}{\sqrt{2\pi}} \left[\sum_{N<|n|\leq M} |\widehat{f^{(p)}}(n)|^2 \right]^{1/2} \left[\sum_{N<|n|\leq M} \frac{1}{|n|^{2p}} \right]^{1/2} \\
&\leq \frac{1}{\sqrt{2\pi}} \|f^{(p)}\|_{L^2} \times \left[2 \int_N^\infty \frac{1}{r^{2p}}\, dr \right]^{1/2} \\
&\leq \frac{\|f^{(p)}\|_{L^2}}{p\sqrt{2\pi} N^{p-1/2}}
\end{aligned}$$

■

8.3 A Sobolev Embedding Theorem

As an application of the foregoing we discuss the simplest example of a Sobolev Embedding Theorem. Recall the we defined the Sobolev space $H^1(T)$ as the Hilbert space obtained by completion of $C^1(T)$ for the norm derived from the inner product

$$\langle f, g\rangle_{H^1} = \langle f, g\rangle_{L^2} + \langle f', g'\rangle_{L^2}$$

Using the the orthonormality of the Fourier basis and (8.6), we can rewrite this as

$$\langle f, g\rangle_{H^1} = 2\pi \sum_{n=-\infty}^{+\infty} \overline{\hat{f}(n)}\hat{g}(n) + \overline{\widehat{f'}(n)}\widehat{g'}(n)$$

In other words, $H^1(T)$ is also the completion of the trigonometric polynomials for the H^1 norm, and, therefore

$$H^1(T) = \{ \sum_{n=-\infty}^{+\infty} a_n e^{inx} \mid \sum_{n=-\infty}^{+\infty} n^2 |a_n|^2 < \infty \}$$

More generally, for $p \geq 1$, one defines

$$H^p(T) = \{ \sum_{n=-\infty}^{+\infty} a_n e^{inx} \mid \sum_{n=-\infty}^{+\infty} n^{2p} |a_n|^2 < \infty \} \quad .$$

Theorem 8.3 (A Sobolev Embedding Theorem) *$H^p(T) \subset C^{p-1}(T)$.*

Proof: In the same way as in the proof of Theorem 8.2, one shows that the condition

$$\sum_{n=-\infty}^{+\infty} n^{2p} |a_n|^2 < \infty$$

implies that the partial sums for $f^{(p-1)}$ converge uniformly. And, hence, the limit is continous. ■

8.4 Multiple Fourier series

Let $n \geq 1$, and denote by T^n the $n-$dimensional torus, i.e., the Cartesian product of n circles. The notion of Fourier series is straightforwardly extended to functions of n variables $f : T^n \to \mathbb{C}$, by considering the o.n. basis of $L^2(T^n)$ consisting of the functions

$$e_{\mathbf{n}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} e^{i\mathbf{n}\cdot\mathbf{x}}, \mathbf{n} \in \mathbb{Z}^n$$

Where $\mathbf{x} = (x_1, \ldots, x_n) \in T^n$ and $\mathbf{n} = (n_1, \ldots, n_n) \in \mathbb{Z}^n$, and $\mathbf{n} \cdot \mathbf{x} = n_1 x_1 + \cdots + n_n x_n$.

Exercise 8.4 *Consider $f : T^3 \to \mathbb{C}$ defined by*

$$f(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} a_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{x}}$$

Where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{n} = (n_1, n_2, n_3)$, and $\mathbf{n}\cdot\mathbf{x} = n_1x_1 + n_2x_2 + n_3x_3$. Prove that if, for some $\alpha > 3$,

$$\sum_{\mathbf{n}\in\mathbb{Z}^3} \|n\|^{\alpha} |a_{\mathbf{n}}|^2 < \infty$$

then, f is continuous.