# Additional Notes on Permutations

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## 1 Introduction

A *permutation* of a list of $n$ objects is a reordering of that list. To describe the reordering itself, the nature of the objects is irrelevant. E.g., we could interchange the second and third item in a list of five objects, no matter what those items are, and this defines a particular permutation that we could perform on any list of five objects.

As the nature of the objects is immaterial for the definition of a permutation, one commonly uses the integers $1, 2, \ldots, n$, as the standard list of $n$ objects. If you like you can think of the integers as labels for the items in any list of $n$ elements.

## 2 Definition

Mathematically, we define a permutation as a transformation of the set $\{1, \ldots, n\}$ into itself.

**Definition 2.1** *A permutation of $n$ elements is a one-to-one and onto map of the set $\{1, 2, \ldots, n\}$ into itself.*

So, a permutation is a map $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, such that for every integer $i$, $1 \leq i \leq n$, there exists exactly one integer $j$, $1 \leq j \leq n$, such that $\pi(j) = i$. We will commonly denote permutations by Greek letters such as $\pi$ (pi), $\sigma$ (sigma), $\alpha$ (alpha), $\beta$ (beta), $\gamma$ (gamma), etc.. The set of all permutations of $n$ elements is denoted by $\mathcal{S}_n$.

**Example:** For $n = 2$, there is only 1 non-trivial permutation, namely the exchange of the 1st and the 2nd element. According to the definition of permutation, the identity map, which is the reordering that does not change the order at all, is also a permutation, called the *trivial* or *identity* permutation. This is analogous to considering adding zero to a number, as a valid addition. Thus, there are two permutations of two elements which reorder the list $1, 2$ as $1, 2$ or $2, 1$. Considered as a map, the nontrivial permutation of two elements is defined by

$$\pi(1) = 1, \qquad \pi(2) = 1 \quad .$$

It is not difficult to find the number of permutations of $n$ elements, i.e., the number of elements of the set $\mathcal{S}_n$. To select a permutation of $n$ elements we can proceed as follows. First, choose an integer $i$, $1 \leq i \leq n$, to put in the first position. Clearly, there are $n$ possible choices. Next, choose the element to go in the second position. After the choice of the first, there are $n - 1$ remaining choices, etc. Therefore we see that

$$\#\mathcal{S}_n = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 = n!$$

**Example:** For $n = 3$, there are $3! = 1 \cdot 2 \cdot 3 = 6$ permutations. A simple way to describe a permutation is to show its effect on the list $1, 2, \ldots, n$. In our example $n = 3$, so an arbitrary permutation of three elements, denoted by $\pi$, can be presented as follows:

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ \pi(1) & \pi(2) & \pi(3) \end{pmatrix} \quad .$$

which, to save space, is often written as

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \quad \text{or} \quad \pi = (\, \pi_1 \quad \pi_2 \quad \pi_3 \,)\,.$$

The 6 permutations of three elements are

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Once more, you should think of a permutation as a reordering operation. E.g., the second permutation given above can be read as defining the reordering which, with respect to the original list places the 2nd element in the 1st position, the 3rd element in the 2nd position, and the 1st in the 3rd position. The same permutation could equally well have been identified by describing its action on the list of the $a, b, c$:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$$

no matter what the letters $a, b, c$ stand for. In particular we also have

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Although these representations of permutations of $n$ elements are $2 \times n$ matrices, at this moment, permutations are not linear transformations of a vector space, and the composition of permutations does *not* correspond to multiplication of these $2 \times n$ matrices.

# 3 Composition of permutations

We have defined permutations as from $\{1, 2, \ldots, n\}$ to itself. Therefore, we can compose two such maps by performing one after the other. As each permutation is a reordering, the result of performing two permutations in succession will be another reordering, i.e., a permutation itself. This means that for any pair $\pi, \sigma \in \mathcal{S}_n$, the composition $\pi \circ \sigma$ (read: *pi after* $\sigma$) is also an element of $\mathcal{S}_n$. The trivial permutation (corresponding to the identity map) is denoted by id and, as it leaves the order unchanged, we have

$$\pi \circ \mathrm{id} = \pi, \quad \text{and} \quad \mathrm{id} \circ \pi = \pi,$$

for all permutations $\pi$. As always, the composition of maps is associative:

$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma) = \alpha \circ \beta \circ \gamma$$

for all permutations $\alpha, \beta$ and $\gamma$.

As each permutation is one-to-one and onto, there is a unique inverse map, i.e., a unique permutation $\sigma$ such that

$$\pi \circ \sigma = \mathrm{id}, \quad \text{and} \quad \sigma \circ \pi = \mathrm{id},$$

$\sigma$ is called the *inverse* of $\pi$ and denoted $\pi^{-1}$. It is the permutation that undoes the reordering performed by $\pi$.

The properties of the composition permutations discussed above are summarized by the statement that $\mathcal{S}_n$ is a *group*. Note that the composition of permutations is not commutative. If $n \geq 3$, you can easily find examples of permutations $\pi$ and $\sigma$ such that $\pi \circ \sigma \neq \sigma \circ \pi$.

# 4 Inversions and the sign of a permutation

For a given permutation $\pi$ we can count the number of inversions it introduces in the ordering of $n$ objects. An *inversion* is a pair $i, j$, such that $i < j$ and $\pi(i) > \pi(j)$. In other words, an inversion is a pair of objects in the list of which the relative order is changed by the permutation. The number of inversions is then just the number of such pairs. If the permutation is represented by its action on $1, 2, \ldots, n$, an inversion is just a pair that appears in decreasing order in the list $\pi_1, \pi_2, \ldots, \pi_n$.

**Example:** The inversions in the 6 permutations of $1, 2, 3$ are indicated by brackets:

$$\begin{array}{ccc}
\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \\
\text{no inversions} & \text{2 inversions: 2,1; 3,1} & \text{2 inversions: 3,1; 3,2} \\
\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \\
\text{1 inversion: 3,2} & \text{3 inversions: 3,2; 2,1; 3,1} & \text{1 inversion: 2,1}
\end{array}$$

One of the simplest nontrivial permutations is one that exchanges the positions of two elements. Such permutations are called *transpositions*, e.g.,

$$t_{12} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

is a transposition. In general, for any $i \neq j$, we will denote the transposition that exchanges $i$ and $j$, by $t_{ij}$. It is an easy exercise to show that the number of inversions in $t_{ij}$ equals $2|i - j| - 1$. So the number of inversions in a transposition is always odd.

**Definition 4.1** *Let $\pi$ be a permutation. Then the sign of $\pi$, denoted by $sign(\pi)$ is defined by*

$$sign\pi = \begin{cases} +1 & \textit{if the number of inversions in } \pi \textit{ is even} \\ -1 & \textit{if the number of inversions in } \pi \textit{ is odd} \end{cases}$$

*The permutation $\pi$ is called even if sign$(\pi) = +1$, and odd if sign$(\pi) = -1$.*

In the example above the sign of the permutations on the first line is $+1$, and the sign of the permutations on the second line is $-1$. As the number of inversions in a transposition is always odd, all transpositions are odd.

It is a general fact that the number of even permutations of $n$ elements is the same as the number of odd permutations, i.e., both numbers are equal to $n!/2$.

The sign has the following properties:

$$\begin{aligned}
&\mathrm{sign}(\mathrm{id}) = 1 \\
&\mathrm{sign}(t_{ij}) = -1, \qquad \text{for all } i \neq j \\
&\mathrm{sign}(\pi \circ \sigma) = \mathrm{sign}(\pi)\mathrm{sign}(\sigma) \\
&\mathrm{sign}(\pi^{-1}) = \mathrm{sign}(\pi)
\end{aligned}$$

# 5 Summations indexed by the set of all permutations

Recall that the determinant of an $n \times n$ matrix $A = (a_{ij})$, is defined by the formula

$$\det(A) = \sum_{\pi} \mathrm{sign}(\pi) a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)} \quad ,$$

where the sum is over all permutations of $n$ elements. To compute $\det(A)$ using this formula we have to add $n!$ terms that are each a product of $n$ factors. $n!$ is a very rapidly increasing function of $n$. E.g., $10! = 3,628,800$. Fortunately, by using some properties of the determinant (listed in the next section) this computation can be greatly reduced in size. The main tool needed to derive these properties are some properties of summations over all permutations, which are in turn based on properties of permutations and the fact that the addition and multiplication of real numbers are commutative.

Suppose $T(\pi)$ is a real number for each permutation $\pi$. E.g., $T(\pi)$ could be the term corresponding to the permutation $\pi$ in the sum that defines the determinant of $A$. To compute

$$\sum_{\pi} T(\pi)$$

we could use any ordering of the set of all permutations, because the sum is independent of the order in which the terms are added. As long as all terms are added exactly once to the sum, the total will be the same. Some commonly used reorderings of such sums are the following.

$$\begin{aligned}\sum_{\pi} T(\pi) &= \sum_{\pi} T(\sigma \circ \pi) \\ &= \sum_{\pi} T(\pi \circ \sigma) \\ &= \sum_{\pi} T(\pi^{-1})\end{aligned}$$

where $\sigma$ is a fixed permutation. These equalities are justified by the following facts. If $\sigma$ is a fixed permutation and $\pi$ runs through all permutations, then $\sigma \circ \pi$ runs through permutations as well, only in a different order. I.e., the action of $\sigma$ is merely to permute the permutations! The same holds for $\pi \circ \sigma$ and $\pi^{-1}$. Another way of saying this is that there is a one-to-one correspondence between permutations and their inverses, which is quite obvious, as every permutation has a unique inverse.

# 6 Properties of the determinant

The following properties of the determinant are given in the textbook by Kolman. Their proofs involve the properties of permutations, of the sign of permutations, and of sums over permutations discussed in the previous sections.

**Theorem 6.1** *Let $A$ and $B$ be $n \times n$ matrices. Then*

$$\det(A^T) = \det(A)$$
$$\det(AB) = \det(A)\det(B)$$

*The determinant of the zero matrix is 0, and the determinant of the identity matrix is 1.*

**Theorem 6.2** *If the matrix $A$ has two identical rows, or two identical colum-s, then* $\det(A) = 0$. *If $A$ has a row or a column consisting entirely of zeroes, then* $\det(A) = 0$.

The effect of ERO's on the determinant is given in the following theorem.

**Theorem 6.3** *Let $A_1$ be the matrix obtained from $A$ by interchanging rows $s$ and $r$, for $s \neq r$. Then*

$$\det(A_1) = -\det(A).$$

*Let $A_2$ be the matrix obtained from $A$ by multiplying row $r$ by a constant $c$. Then*

$$\det(A_2) = c\det(A).$$

*Let $A_3$ be the matrix obtained from $A$ by replacing row $s$ by the sum of row $s$ and $d$ times row $r$. Then*

$$\det(A_3) = \det(A)$$

These properties of ERO's in combination with the following theorem, give an alternative way of computing determinants of relatively large matrices.

**Theorem 6.4** *If $A$ is upper or lower triangular, we have*

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$