Take-home Final Exam. Due: Thursday, 11 December 2003, 6pm

Choose four problems to turn in. I will not grade more than four problems for your final exam grade.

Problem 1. Let $\mathcal{H}$ be a finite-dimensional one-particle Hilbert space, and $\mathcal{F}^-$ the corresponding fermionic Fock space. Prove that

$$\dim \mathcal{F}^- = 2^{\dim \mathcal{H}}.$$

Problem 2. Show that for any one-particle Hilbert space (i.e., any Hilbert space $\neq \{0\}$) the bosonic Fock space, $\mathcal{F}^+$, is infinite-dimensional.

Problem 3. Let $\mathcal{H}$ be a one-particle Hilbert space and let $\mathcal{F}^-$ be the corresponding fermionic Fock space. Verify the Canonical Anticommutation Relations (CAR), i.e., for all $\phi, \psi \in \mathcal{H}$, one has

$$\{a(\phi), a^*(\psi)\} = (\phi, \psi)\mathbb{1}, \quad \{a^*(\phi), a^*(\psi)\} = 0, \quad \{a(\phi), a(\psi)\} = 0,$$

where, $\{A, B\} = AB + BA$, is the *anticommutator* of A and B.

Problem 4. Let $\mathcal{H}$ be a one-particle Hilbert space and let $\mathcal{F}^-$ be the corresponding fermionic Fock space. Show that for every $\phi \in \mathcal{H}$, $a(\phi)a^*(\phi)$ is self-adjoint with only discrete spectrum, which is the set $\{0, \|\phi\|^2\}$.

Problem 5. Let $\mathcal{H}$ be a separable Hilbert space and suppose $\{e_k\}_{k\geq 0}$ is an o.n. basis for $\mathcal{H}$. Prove that on the bosonic and fermionic Fock spaces with one-particle space $\mathcal{H}$, we have the following expression for the number operator:

$$N = \sum_{k=0}^{\infty} a^*(e_k)a(e_k).$$

Problem 6. Let $\mathcal{H}$ be a one-particle Hilbert space and let $\mathcal{F}^-$ be the corresponding fermionic Fock space. Consider transformations U and V, $\mathcal{H} \to \mathcal{H}$, and define

$$\tau(a(\phi)) = a(U\phi) + a^*(V\phi), \text{ acting on } \mathcal{F}^-.$$

Find sufficient conditions (and also necessary conditions, if you can) on U and V such that τ can be extended to an automorphism of the C^*-algebra generated by $\{a(\phi) \mid \phi \in \mathcal{H}\}$.

Problem 7. Discuss the bosonic version of the previous problem.