Homework #1. Due: Tuesday, 14 October 2003

Choose 3 of the 9 problems to turn in. I won't be able to grade more than 3 problems for your grade on homework #1.

Problem 1. Consider the Pauli matrices defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

a) Verify that

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \sum_{k=1}^{3} \epsilon_{ijk} \sigma_k$$

where, for $i, j, k \in \{1, 2, 3\}$, ϵ_{ijk} is the Levi-Civita symbol, which is 0, whenever two or more of the indices are equal, and otherwise equal to the sign (± 1) of the permutation $(123) \mapsto (ijk)$.

b) For any $\mathbf{x} \in \mathbb{C}^3$, define $\mathbf{x} \cdot \sigma = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, show that

$$(\mathbf{x} \cdot \sigma)(\mathbf{y} \cdot \sigma) = \mathbf{x} \cdot \mathbf{y} \mathbb{1} + i(\mathbf{x} \times \mathbf{y}) \cdot \sigma$$

c) Prove that the set of eigenvalues of $\mathbf{n} \cdot \sigma$ is $\{1, -1\}$, for all unit vectors $\mathbf{n} \in \mathbb{R}^3$.

Problem 2. Let the Pauli matrices be defined as in the previous problem, and suppose

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is a power series with radius of convergence $R > 0$. Prove that for any $\mathbf{x} \in \mathbb{C}^3$, with $0 < \|\mathbf{x}\| = r < R$, one has

$$f(\mathbf{x} \cdot \sigma) = \frac{1}{2}(f(\|\mathbf{x}\|) + f(-\|\mathbf{x}\|))\mathbb{1} + \frac{1}{2}(f(\|\mathbf{x}\|) - f(-\|\mathbf{x}\|))\frac{\mathbf{x}}{\|\mathbf{x}\|} \cdot \sigma$$

You can assume the results of Problem 1 in your derivation.

Problem 3. Consider the real linear space, V, of traceless 2×2 complex matrices, for which the Pauli matrices are a basis. For any unitary 2×2 matrix with determinant 1, $U \in SU(2)$, define the linear map $\pi_U : V \to V$, by

$$\pi_U(\mathbf{x} \cdot \sigma) = U^* \mathbf{x} \cdot \sigma U, \mathbf{x} \in \mathbb{R}^3 \quad .$$

Find the map $R : U(2) \to M_3(\mathbb{R})$ such that

$$\pi_U(\mathbf{x} \cdot \sigma) = (R(U)\mathbf{x}) \cdot \sigma$$

Problem 4.

a) Find the states $\psi \in \mathbb{C}^2$, with the property that a measurement of σ_j, will have outcome $+1$ with probability $1/2$, for $j = 1, 2, 3$, i.e., no matter which component you choose to measure, you will find $+1$ with probability $1/2$.

b) Does the property of the states ψ found in part a) imply that measurement of *any* component of σ, say in the $\mathbf{n}$-direction, will yield $+1$ with probability $1/2$? Why or why not?

Problem 5. Define three units vector sin $\mathbb{R}^3$ as follows:

$$\mathbf{n}_1 = (0, 0, 1), \quad \mathbf{n}_2 = (\sin\theta, 0, \cos\theta), \quad \mathbf{n}_3 = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha),$$

and consider a sequence of three Stern-Gerlach experiments, which measure the $\mathbf{n}_1$-, $\mathbf{n}_2$-, and $\mathbf{n}_3$-components of a beam of spin 1/2 particles, in that order.

a) Calculate the probabilities of finding each of following sequences of outcomes: $(+1, +1, +1)$, $(+1, +1, -1)$, $(+1, -1, +1)$.

b) What is the the probability of measuring $(+1, +1 \text{or} - 1, +1)$?

c) Compare the result of b) with the probability of finding $(+1, +1)$, in an experiment where only the $\mathbf{n}_1$-, and $\mathbf{n}_3$-components are measured.

Problem 6. Consider a tree of Stern-Gerlach instruments constructed as follows. For a positive integer $N \geq 1$, and $k = 0, \ldots, N$, define

$$\mathbf{n}(k) = (\sin k\pi/(2N), 0, \cos k\pi/(2N))$$

A beam of silver atoms enters the first apparatus, which measures the $\mathbf{n}(0)$-component of the magnetic moment and splits the beam in two parts according to the results. These two beams then enter each another Stern-Gerlach instrument which is set to measure the $\mathbf{n}(1)$-component. Next, four instruments are measuring the $\mathbf{n}($2-components of each of the four beams leaving the previous experiment. At the end we have 2^{N+1} beams, labeled by a sequence of plus and minus one's, the outcomes of the successive measurements of $\mathbf{n}(0), \mathbf{n}(1), \ldots, \mathbf{n}(N)$. For $b \in \{1, -1\}^{N+1}$, let $P(b)$ denote the probability that a silver atom ends up in beam b.

a) Suppose the initial state of each atom in the beam is $\psi_0 = |z, +\rangle$, the eigenstate of σ_3 with eigenvalue 1. Calculate $\max_b P(b)$, and find the b where the maximum it is attained.

b) What happens in the limit $N \to +\infty$?

Problem 7. Define the Hermite polynomials H_n by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2} \right).$$

a) Show that

$$\phi_n(x) = e^{-x^2/2} H_n(x)$$

is an orthogonal set in $L^2(\mathbb{R})$.

b) Show that the nth Hermite function ϕ_n is an eigenfunction of the linear operator

$$H = -\frac{d^2}{dx^2} + x^2$$

with eigenvalue

$$\lambda_n = 2n + 1.$$

Hint: Let

$$A = \frac{d}{dx} + x, \qquad A^* = -\frac{d}{dx} + x.$$

Show that

$$A\phi_n = 2n\phi_{n-1}, \quad A^*\phi_n = \phi_{n+1}, \quad H = AA^* - 1.$$

c) Note that H is the *Hamiltonian operator* of a harmonic oscillator. A^* and A (or closely related operators) are called *creation* and *annihilation*, or *ladder*, operators. Define domains for these unbounded operators on $L^2(\mathbb{R})$, and prove the algebraic relations of part b) for the unbounded operators.

Problem 8. For $\psi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, prove the formula

$$e^{-i\frac{t}{2}\Delta}\psi(x) = \left(\frac{2\pi i t}{m}\right)^{-d/2} \int_{\mathbb{R}^d} e^{\frac{im}{2t}|x-y|^2}\psi(y)\, dy \quad .$$

Problem 9. For $\psi \in L^2(\mathbb{R}^n)$, define the *Wigner distribution* of ψ, $W(x,k)$, for $x, k \in \mathbb{R}^n$, by

$$W(x,k) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \psi\left(x - \frac{y}{2}\right) \overline{\psi\left(x + \frac{y}{2}\right)} e^{ik\cdot y}\, dy.$$

a) Compute the Wigner distribution of a Gaussian $\exp(-x \cdot Ax)$, where A is a positive definite matrix.

b) Show that W is real-valued, and

$$W(x,k) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{\psi}\left(k - \frac{\ell}{2}\right) \overline{\hat{\psi}\left(k + \frac{\ell}{2}\right)} e^{-i\ell\cdot x}\, d\ell,$$

$$\int_{\mathbb{R}^n} W(x,k)\, dk = |\psi(x)|^2, \qquad \int_{\mathbb{R}^n} W(x,k)\, dx = \left|\hat{\psi}(k)\right|^2,$$

where $\hat{\psi}$ denotes the Fourier Transform of ψ.

c) From part b) we see that the Wigner distribution W has some properties of a joint distribution for position and momentum in the state ψ, $(x,k) \in \mathbb{R}^{2n}$, the classical phase space. Show, however, that W is not necessarily nonnegative.