Homework #2. Due: Tuesday, 28 October 2003

Choose 3 problems to turn in. I won't be able to grade more than 3 problems for your grade on homework #2.

Problem 1. Let $\psi \in L^2(\mathbb{R})$ be such that

$$\Delta_x^2 = \int_{-\infty}^{\infty} x^2 |\psi(x)|^2 \, dx < +\infty, \qquad \Delta_k^2 = \int_{-\infty}^{\infty} k^2 |\hat{\psi}(k)|^2 \, dk < +\infty,$$

where $\hat{\psi}$ denotes the Fourier Transform of ψ, and

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 1.$$

a) Prove the *Heisenberg uncertainty principle* by using properties of the Fourier Transform:

$$\Delta_x^2 \Delta_k^2 \geq \frac{1}{4}.$$

b) Show that equality is attained when ψ is a suitable Gaussian.

Problem 2. Consider the Hamiltonian

$$H = -\frac{1}{2m} \frac{d}{dx^2} + \lambda x^4$$

defined on a suitable domain in $L^2(\mathbb{R})$, and with $m, \lambda > 0$. Find a strictly positive lower bound for the smallest eigenvalue of H. If necessary, you may assume that H has an o.n. basis of eigenvectors $e_n \in L^2(\mathbb{R})$.

Problem 3. For all integers $n \geq 1$, prove that there is a one-to-one correspondence between states on M_n and $n \times n$ density matrices.

Problem 4. Let ω be a state on M_2, and define real numbers a, b, r by

$$\omega(\sigma_1) = 2a, \quad \omega(\sigma_2) = 2b, \quad \omega(\sigma^3) = 2r - 1. \tag{1}$$

a) Show that $0 \leq r \leq 1$ and $a^2 + b^2 \leq r(1 - r)$.

b) Show that for any triple (a, b, r) of real numbers satisfying the properties of part a), there is exactly one state ω on M_2 such that the relations (1) hold.

Problem 5.

a) For every $\psi \in \mathbb{C}^n$, with $\|\psi\| = 1$, prove that $\omega(A) = \langle \psi, A\psi \rangle$, for all $A \in M_n$, defines a pure state on M_n.

b) If ω is a *pure* state on M_n, prove that there exists $\psi \in \mathbb{C}^n$ such that $\omega(A) = \langle \psi, A\psi \rangle$, for all $A \in M_n$.

c) Show that the correspondence between pure states ω and vectors ψ is one-to-one up to a phase factor, i.e., show that if ψ and ϕ define the same state ω, then there is $\alpha \in \mathbb{R}$ such that $\phi = e^{i\alpha}\psi$.

Problem 6. Let ρ be an $n \times n$ density matrix, and let ω be the state on M_n determined by ρ. Let $e_1, \ldots, e_n$ be an o.n. basis of $\mathbb{C}^n$, and $\rho_1, \ldots, \rho_n \in \mathbb{R}$, such that $\rho e_i = \rho_i e_i$, $1 \leq i \leq n$.

a) Suppose $\rho_i > 0$, for all $i = 1, \ldots, n$. Prove that the GNS representation of ω (up to unitary equivalence) is given by the triple $(\mathcal{H}, \pi, \Omega)$, with the following definitions: $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^n$, $\pi(A) = A \otimes \mathbb{1}, A \in M_n$, and $\Omega = \rho_i^{1/2} e_i \otimes e_i$.

b) What is the GNS representation of ω, if ρ has zero eigenvalues?

Problem 7. Let η be the state on M_n determined by the density matrix $n^{-1}\mathbb{1}$.

a) Show that $\eta(AB) = \eta(BA)$, for all $A, B \in M_n$.

b) Prove that η is uniquely determined by the property a), i.e., if ω is a state on M_n such that $\omega(AB) = \omega(BA)$, for $A, B \in M_n$, then $\omega = \eta$.

Problem 8. Let $\mathcal{H}$ be a complex Hilbert space and recall the definition of the operator norm of $A \in \mathcal{B}(\mathcal{H})$:

$$\|A\| = \sup\{\|A\psi\| \mid \psi \in \mathcal{H}, \|\psi\| = 1\}.$$

Prove that the operator norm indeed has the C^* property, i.e., show that $\|A^*A\| = \|A\|^2$.

Problem 9. Let $A = A^* \in \mathcal{B}(\mathcal{H})$. Prove the following identity:

$$\|A\| = \sup_{\psi \in \mathcal{H}, \|\psi\|=1} |\langle \psi, A\psi \rangle|.$$

Problem 10. Compute the GNS Hamiltonian for a harmonic oscillator in its equilibrium state at inverse temperature $\beta > 0$, and find its spectral resolution.