Homework #3. Due: Thursday, 13 November 2003

Choose 3 problems to turn in. I won't be able to grade more than 3 problems for your grade on homework #3.

Problem 1. Let A be a linear operator on a Hilbert space H, and $\lambda \in \mathbb{C}$ such that $\dim \ker(A - \lambda) = \infty$. Prove that there exists a Weyl sequence for $A - \lambda$.

Problem 2. Let A be a self-adjoint operator on a Hilbert space H, and $\lambda_n \in \mathbb{C}$ a convergent sequence of eigenvalues of A with limit $\lambda \in \mathbb{C}$. Prove that there exists a Weyl sequence for $A - \lambda$.

Problem 3. Let $\lambda \in \sigma_p(A)$, for a linear operator A on a Hilbert space $\mathcal{H}$. Prove that there does not exists a Weyl sequence for $A - \lambda$

Problem 4. Let A be a linear operator on a Hilbert space $\mathcal{H}$, and $U \in \mathcal{B}(\mathcal{H})$ a unitary. Show that the pure point spectrum and the continuous spectrum of A are both unitarily invariant, i.e.,

$$\sigma_p(U^*AU) = \sigma_p(A), \quad \sigma_c(U^*AU) = \sigma_c(A).$$

Problem 5. Consider the momentum operator P as a self-adjoint operator on $L^2(\mathbb{R})$, with domain $H^1(\mathbb{R})$. Prove that $\sigma(P) = \sigma_c(P) = \mathbb{R}$.

Problem 6. Consider the kinetic energy operator $-\Delta$ as a self-adjoint operator on $L^2(\mathbb{R}^d)$, with domain $H^2(\mathbb{R}^d)$. Prove that $\sigma(-\Delta) = \sigma_c(-\Delta) = [0, +\infty)$.

Problem 7. Consider the Hamiltonian $H = P^2 + X^2$ of the one-dimensional harmonic oscillator as a self-adjoint operator on $L^2(\mathbb{R})$. Prove that $\sigma(H) = \sigma_p(H) = \{2n+1 \mid n \geq 0\}$.

Problem 8. For any linear operator A on a Hilbert space $\mathcal{H}$, and $\lambda \in \rho(A)$, define the resolvent operator, R_λ, by $R_\lambda = (\lambda \mathbb{1} - A)^{-1}$. Prove the *resolvent equation* (also called the *first resolvent identity*):

$$R_\lambda - R_\mu = (\mu - \lambda) R_\mu R_\lambda, \quad \text{for } \lambda, \mu \in \rho(A).$$

Problem 9. Let A be a linear operator on $L^2(\mathbb{R}^d)$, $d \geq 1$, and (ψ_n) a spreading (Weyl) sequence for A. Prove that (ψ_n) is a Weyl sequence for A.

Problem 10. Consider the self-adjoint Schrödinger operator $H = -\Delta + V$, on $L^2(\mathbb{R})$, where V is a confining potential, i.e., $\lim_{|x|\to\infty} V(x) = +\infty$. Let A be a bounded self-adjoint operator on $L^2(\mathbb{R})$. Prove that $\sigma_c(H + A) = \emptyset$.

Problem 11. Let Q and P denote two densely defined self-adjoint operators on a Hilbert space $\mathcal{H}$, such that $[Q, P] = i\mathbb{1}$, on a dense subspace $\mathcal{H}_0 \subset \mathcal{H}$. Prove that both Q and P are unbounded.