Homework #1. Due: Tuesday, 24 January 2006

Problem 1. Consider the Ising model on a ring of L vertices (i.e., the integers $1, \ldots, L$ with addition mod L) with Hamiltonian H defined on $\Omega = \{-1, 1\}^L$ by

$$H(\eta) = -J \sum_{x=1}^{L} \eta_x \eta_{x+1} - B \sum_{x=1}^{L} \eta_x$$

where $J, B \in \mathbb{R}$ are the coupling constant and the external magnetic field, respectively. Recall the transfer matrix

$$T = \begin{pmatrix} e^{\beta(J+B)} & e^{\beta(-J+B)} \\ e^{\beta(-J-B)} & e^{\beta(J-B)} \end{pmatrix}$$

where $\beta \in [0, \infty)$ is the inverse temperature.

Calculate the following quantities:

a) The partition function $Z_L(\beta J, \beta B)$ defined by

$$Z_L(\beta J, \beta B) = \sum_{\eta \in \Omega} e^{-\beta H(\eta)} \, .$$

and the free energy per spin, $f(\beta J, \beta B)$, defined as the limit

$$f(\beta J, \beta B) = \lim_{L \to \infty} -\frac{1}{\beta L} \log Z_L(\beta J, \beta B) \, .$$

b) The one- and two-point functions defined by

$$\omega_L(\sigma_1) = \frac{1}{Z_L(\beta J, \beta B)} \sum_{\eta \in \Omega} \eta_1 e^{-\beta H(\eta)}$$

$$\omega_L(\sigma_1 \sigma_r) = \frac{1}{Z_L(\beta J, \beta B)} \sum_{\eta \in \Omega} \eta_1 \eta_r e^{-\beta H(\eta)} \, , \; 1 \leq r \leq L$$

c) The thermodynamic limits of the $n-$point probability densities:

$$\rho(\eta_1, \ldots, \eta_n) = \lim_{L \to \infty} \text{Prob}(\sigma_1 = \eta_1, \ldots, \sigma_n = \eta_n) \, .$$

Problem 2. Discuss the limits $\beta \to 0$ and $\beta \to \infty$ of the equilibrium states (probability measures) of the Ising model on a ring as a function of J and B.