Homework #2. Due: Tuesday, 7 February 2006

Problem 1. Let $n \geq 1$ and consider $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Let $C_0 \subset \mathbb{R}^n$ denote the *essential domain* of f, i.e., the set of points where f is finite. Recall the definition of f^*, the *Legendre transform* of f:

$$f^* = \sup_{x \in \mathbb{R}^n} \left(\langle y, x \rangle - f(x) \right) .$$

Define the *epigraph* of f, $\mathrm{epi}(f)$, to be the set

$$\mathrm{epi}(f) = \{(x, r) \in C_0 \times \mathbb{R} \mid r \geq f(x)\} .$$

a) Suppose C_0 is convex. Then, show that f is convex iff $\mathrm{epi}(f)$ is a convex subset of $\mathbb{R}^{n+1}$.

b) Prove that for any f, we have $\mathrm{epi}(f^{**}) = \overline{\mathrm{co}}\, \mathrm{epi}(f)$, where $\overline{\mathrm{co}}$ denotes the closed convex hull.

c) Show that if f is convex, then

$$(f^*)^*(x) = f(x) ,$$

for all x in the interior of C_0. (Hint: note that $\overline{\mathrm{co}}\, \mathrm{epi}(f)$ is the intersection of a family of closed halfspaces.)

d) Let $n = 1$ and $f(x) = |x|$. Find f^*.

Problem 2. Consider the one-dimensional Ising model defined on intervals of the form $[-L, L] \subset \mathbb{Z}$ with Hamiltonians $H_L^{b_L}$ defined by

$$H_L^{b_L} = -J \sum_{x=-L}^{L-1} \sigma_x \sigma_{x+1} + b_L$$

where, for each $L \geq 1$, b_L is a function of the boundary spin variables σ_{-L} and σ_L. Fix any inverse temperature $\beta \in \mathbb{R}$. Prove that the sequence Gibbs states at inverse temperature β, ω_L on $C(\Omega_{[-L,L]})$, determined by $H_L^{b_L}$, converges (in the weak-$*$ sense) to a translation invariant state on $C(\Omega_{\mathbb{Z}})$, which depends on J and β, but not on b_L.

Problem 3. Let γ be a non-selfintersecting closed path consisting of nearest neigbor bonds in $\mathbb{Z}^2$ (i.e., a *simple contour*), and denote by $V(\gamma)$ the lattice sites enclosed by γ, and by $l(\gamma)$ the length of γ (i.e., the number of bonds).

a) Prove the following bound:

$$|V(\gamma)| \leq \frac{1}{16} l(\gamma)^2 .$$

b) Consider a finite subset $\Lambda \subset \mathbb{Z}^2$, and let $l = 4, 6, 8, \ldots$. Let $M_\Lambda(l)$ denote the number of simple contours γ of length l contained in Λ. Prove the bound

$$M_\Lambda(l) \leq 3^{l-1}|\Lambda|\,,$$

for all $l = 4, 6, 8, \ldots$.

c) (Optional) Show that there exists $c > 1$ such that for all $l = 4, 6, 8, \ldots$, there exists L_l such that for all $\Lambda = [1, L]^2$, with $L \geq L_l$ one has the bound

$$M_\Lambda(l) \geq L^2 c^l\,.$$

Problem 4. Fix $J > 0$ and $\beta \geq 0$. Prove that the Gibbs states at inverse temperature β of the d-dimensional translation invariant Ising model defined on $[-L, L]^d \subset \mathbb{Z}^d$ with periodic boundary conditions (i.e., defined on tori), converge in the thermodynamic limit $L \to \infty$.

Problem 5. Let $\omega^+_{d,\beta}$ be the limiting Gibbs state of the translation invariant ferromagnetic Ising model on $\mathbb{Z}^d$ with coupling constant $J = 1$, at inverse temperature $\beta \geq 0$. Define the *critical inverse temperature*, β_c, by

$$\beta_c = \sup\{\beta \geq 0 \mid \omega^+_{d,\beta}(\sigma_0) = 0\}\,.$$

Prove that $\beta_c(d)$ is a monotone decreasing function of the dimension d.