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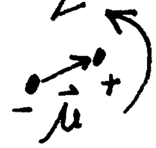
Electromagnetic Interaction is Described by Fields

Electric field $\vec{E} \in \mathfrak{X}(\mathbb{R}^3 \times \mathbb{R})$ $\quad \stackrel{\bullet}{q} \longrightarrow \vec{F} = q\vec{E}$

Magnetic field $\vec{B} \in \mathfrak{X}(\mathbb{R}^3 \times \mathbb{R})$ $\quad \vec{\tau} = \mu \times \vec{B}$



$(\vec{E}, \vec{B}) \in \Lambda_2(\mathbb{R}^3 \times \mathbb{R})$

Maxwell's Equations

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = \rho \\ \vec{\nabla} \times \vec{B} - \dfrac{\partial \vec{E}}{\partial t} = \vec{J} \end{cases} \longleftarrow \text{sources: charge \& current density}$$

$$\begin{cases} \vec{\nabla} \times \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0 \\ \vec{\nabla} \cdot \vec{B} = 0 \end{cases} \longleftarrow \text{no sources (no magnetic manifold)}$$

Note Introducing a metric $ds^2 = -dt^2 + d\vec{x}^2$ on $\mathbb{R}^3 \times \mathbb{R}$

one-form $J = \rho\, dt + \vec{J} \cdot d\vec{x}$

two-form $F = \vec{E}\, dt \wedge d\vec{x} + (\vec{B} \times d\vec{x}) \wedge d\vec{x}$

Then Maxwell's equations read

$$\begin{cases} d{*}F = {*}J \\ dF = 0 \end{cases}$$

Relativistic Quantum Field Theory $\longrightarrow$ QED

Quantum Mechanics	Quantum Field Theory
$\vec{x}(t)$, $\vec{p}(t)$	$\vec{E}(\vec{x},t)$, $\vec{B}(\vec{x},t)$

QFT is infinitely many
copies of QM.

Quantum Mechanics in Heisenberg Picture

Harmonic oscillator $x(t)$, $p(t)$. (operators)

$$\dot{x} = p$$

$$\ddot{x} = -x$$

Solution

$$x = \frac{e^{it} a + e^{-it} a^{+}}{\sqrt{2}}$$

$$p = \frac{i(e^{it} a - e^{-it} a^{+})}{\sqrt{2}}$$

Hamiltonian

$$H = \frac{1}{2}(x^2 + p^2) = \frac{1}{2}(aa^{+} + a^{+}a)$$

time independent.

Schrödinger's equation

$$\dot{x} = \frac{1}{i}[H, x]$$

Holds if $[a, a^{+}] = 1$ because $H = aa^{+} - \frac{1}{2}$

is a counting operator.

The fields, $x(t)$, $p(t)$ satisfy
"equal time commutation relations":

$$[p(t), x(t)] = i$$

States are built on a vacuum $|0\rangle$

$$a|0\rangle = 0$$

by acting with a^{+}

$$(a^{+})^{n}|0\rangle = |n\rangle$$

$$H|n\rangle = (n + \tfrac{1}{2})|n\rangle$$

Repeat same exercise with $\vec{E}$ and $\vec{B}$.

Step ① Want to solve Maxwell's equations. Study free case $\rho = \vec{J} = 0$.

First consider the pair

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

These have a trivial solution by choosing better variables

$$\vec{E} = +\frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = -\vec{\nabla} \times \vec{A}$$

because $\nabla \cdot \nabla \times \vec{V} \equiv 0$ identically

$$\nabla \times \vec{E} = +\frac{\partial}{\partial t}(\nabla \times \vec{A}) = -\frac{\partial \vec{B}}{\partial t}$$

The new field $\vec{A}(\vec{x},t)$ is called a <u>vector potential</u> and will play the rôle of $x(t)$.

Hence $\vec{E} = \dot{\vec{A}}$ is the analogue of $p(t)$.

Note that replacing

$$\vec{A} \to \vec{A} + \vec{\nabla}\alpha$$

is also a solution. This is called a "residual gauge invariance" which we fix by requiring

$$\nabla \cdot \vec{A} = 0$$

(just choose $\Delta\alpha = -\nabla \cdot \vec{A}$). Gauss Constraint.

As a corollary we have solved $\vec{\nabla} \cdot \vec{E} = 0$, since

$$\nabla \cdot \vec{E} = \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) = 0 \quad \forall t.$$

Finally examine

$$0 = -\frac{\partial \vec{E}}{\partial t} + \nabla \times \vec{B} = -\frac{\partial^2 \vec{A}}{\partial t^2} + \vec{\nabla} \times (\vec{\nabla} \times \vec{A})$$

$$= \left(-\frac{\partial^2}{\partial t^2}\vec{A} + \Delta \vec{A}\right) - \vec{\nabla}\,\cancelto{0}{\vec{\nabla} \cdot \vec{A}} \quad \text{Wave eq}^{\underline{n}}$$

Summary $\vec{E} = +\dot{\vec{A}}$, $\vec{B} = -\nabla \times \vec{A}$ where

$$\left(-\frac{\partial^2}{\partial t^2} + \Delta\right)\vec{A} = 0$$

$$\nabla \cdot \vec{A} = 0$$

Solution

$$\vec{A}(\vec{x},t) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}} \left(\vec{a}_{\vec{k}}\, e^{i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}} + \vec{a}^{\dagger}_{\vec{k}}\, e^{-i\omega_{k} - i\vec{k}\cdot\vec{x}} \right)$$

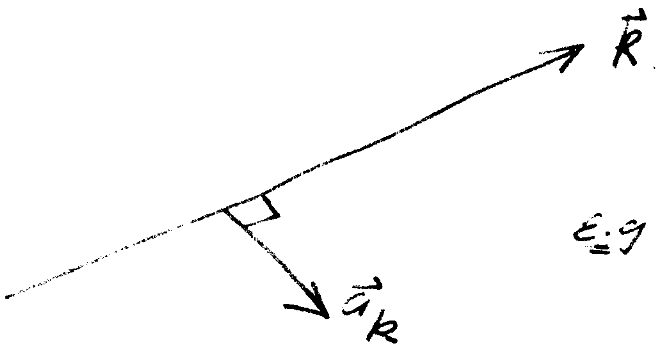
} Complex Fourier coefficients

$\omega_{\vec{k}} \equiv \sqrt{\vec{k}^2}$ ⟶ DISPERSION RELATION FOR PHOTONS

$$k_\mu = (\omega_{\vec{k}}, \vec{k})$$

$$0 = -\omega_{\vec{k}}^2 + \vec{k}^2 = k_\mu \eta^{\mu\nu} k_\nu \quad \text{light-like}$$

Require $\quad \vec{k}\cdot\vec{a}_{\vec{k}} = 0 = \vec{k}\cdot\vec{a}^{\dagger}_{\vec{k}}$ "Plaritations"



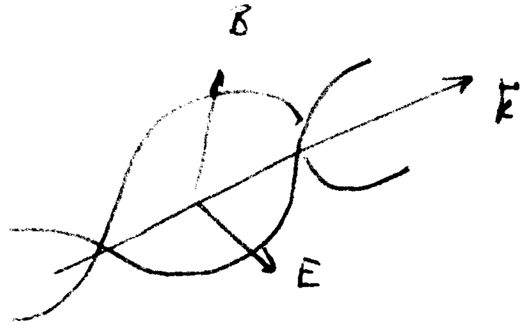
e.g. $\vec{k} = (0,0,k_z)$

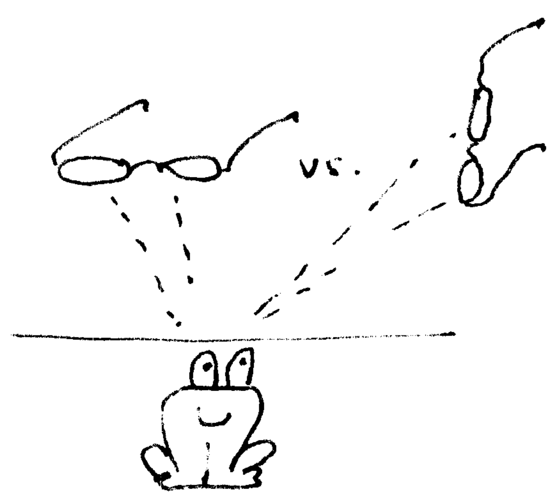
$$\vec{a}_k = (\vec{\varepsilon}_x + i\vec{\varepsilon}_y)\, a_k$$

$$\vec{\varepsilon}_x = (1,0,0)$$

$$\vec{\varepsilon}_y = (0,1,0)$$

"circularly polarized light".





Want to mimic Harmonic oscillator analysis, but now have infinitely many operators $\vec{a}_{\vec{k}}$ labeled by 2 possible polarizations & $\vec{k} \in \mathbb{R}^3$.

$$\vec{A} = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}} \left(e^{ik\cdot x}\vec{a}_{\vec{k}} + e^{-ik\cdot x}\vec{a}_{\vec{k}}^{+}\right)$$

$$\vec{E} = i\int \frac{d^3k\,\omega_k}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}} \left(e^{ik\cdot x}\vec{a}_{\vec{k}} - e^{-ik\cdot x}\vec{a}_{\vec{k}}^{+}\right)$$

Energy

$$E = \int d^3x\, \tfrac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right)$$

$$= \int d^3k\, \omega_{\vec{k}} \left(\frac{\vec{a}_{\vec{k}}\cdot\vec{a}_{\vec{k}}^{+} + \vec{a}_{\vec{k}}^{+}\cdot\vec{a}_{\vec{k}}}{2}\right)$$

easy calculation

Hence postulating $\vec{a}_{\vec{k}}, \vec{a}_{\vec{k}}^{+}$ to be operators satisfying

$$[\vec{a}_{\vec{k}}, \vec{a}_{\vec{k}'}] = \delta^3(\vec{k}-\vec{k}')\left(\mathbb{1} - \frac{\vec{k}\otimes\vec{k}'}{\vec{k}^2}\right)$$

find equal time canonical commutation relations

$$[\vec{E}(\vec{x},t), \vec{A}(\vec{x}',t)] = \delta^3(\vec{x}-\vec{x}')\left(\mathbb{1} - \frac{\vec{\nabla}\otimes\vec{\nabla}}{\Delta}\right)$$

again easy Note $\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} = \delta^3(\vec{x})$.

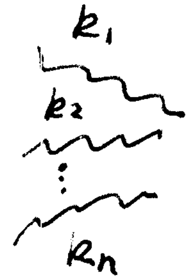
States in the theory – Fock space

$$\text{vacuum} \quad |0\rangle \,, \quad \vec{a}_k |0\rangle \equiv 0$$

$$\vec{a}^{\,+}_{\vec{k}} |0\rangle = \vec{\varepsilon}\, a^{+}_{\vec{k}} |0\rangle = |\vec{\varepsilon}(\vec{k}), \vec{k}\rangle$$

single photon state: 

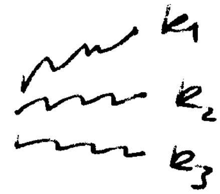
Multiphoton state:

$$\vec{a}^{\,+}_{\vec{k}_1} \vec{a}^{\,+}_{\vec{k}_2} \cdots \vec{a}^{\,+}_{\vec{k}_n} |0\rangle$$ 

Important: Quantum mechanics describes only fixed numbers of particles.

Quantum Field theory includes production & annihilation

$|in\rangle =$ 

$\langle out| =$ 

"second quantization"

Notice also the QFT folklore

fields $\longleftrightarrow$ particles.

Energy of vacuum

$$E = \int d^3k\, \omega_{\vec{k}} \left(\vec{a}^{\dagger}_{\vec{k}} \vec{a}_{\vec{k}} + \frac{1}{2} \right)$$

$$E|0\rangle = \frac{1}{2} \int d^3k\, \omega_{\vec{k}} \,|0\rangle$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} dk_1 dk_2 dk_3 \sqrt{k_1^2 + k_2^2 + k_3^2}\, |0\rangle$$

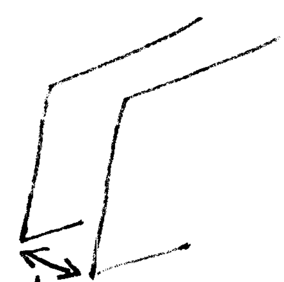
INFINITY

First example of divergence in QFT.

→ In general require renormalization

→ Here not serious, just adds ∞ constant to the energy → "normal ordering prescription"

→ The effect is real, can do same calculation with boundary conditions



Pair of parallel plates

compute $E(d)$

recall

$$\frac{\partial V}{\partial x} = -F$$

this force can be calculated (including radiative corrections) and measured → astounding accuracy!

Interactions

Everything above was <u>free fields</u>

Really want light interacting with charged matter, e.g. electrons

electrons $\longleftrightarrow$ Fermi fields
(Grassmann odd)
$\{b_{\vec{k}}, b^+_{\vec{k}}\} = \delta(\vec{k}-\vec{k}')$
(can't have two fermions in same state $(b^+_{\vec{k}})^2 = 0$)

<u>Action</u> $S_{QED} = -\frac{1}{4}\int d^4x\, F^2_{\mu\nu} - \int d^4x\, \bar{\psi} \not{D} \psi$

$\not{D} = \gamma^\mu(\partial_\mu + ie A_\mu)$

$S_{int} = -ie \int d^4x\, \bar{\psi}\gamma^\mu A_\mu \psi$

<u>S-matrix</u> $\langle out | e^{iHt} | in \rangle$

<u>Famous examples</u>

Compton scattering

Calculable and measured to 9 or 10 digit <u>accuracy</u>.

scattering of light by light