MAT 21C: PRACTICE PROBLEMS LECTURE 11

PROFESSOR CASALS (SECTIONS B01-08)

Abstract. Practice problems for the eleventh lecture of Part II, delivered May 31 2023. Solutions will be posted within 48h of these problems being posted.

Problem 1. Evaluate the following functions $f(x, y)$ at the indicated points (x, y).

(a) Evaluate $f(x, y) = e^{xy} \cos(x + y)$ at $(0, 0)$, $(\pi, 0)$, $(0, \pi)$ and (π, π).
We can evaluate the function by plugging in the x and y values into $f(x, y)$ as follows. First, we find that $f(0, 0)$ is

$$f(0, 0) = e^{0 \cdot 0} \cos(0 + 0) = 1 \cdot 1 = 1.$$

Then, we find that $f(\pi, 0)$ is

$$f(\pi, 0) = e^{\pi \cdot 0} \cos(\pi + 0) = 1 \cdot -1 = -1.$$

Then, we find that $f(0, \pi)$ is

$$f(0, \pi) = e^{0 \cdot \pi} \cos(0 + \pi) = 1 \cdot -1 = -1.$$

Finally, we find that $f(\pi, \pi)$ is

$$f(\pi, \pi) = e^{\pi \cdot \pi} \cos(\pi + \pi) = e^{\pi^2} \cos(2\pi) = e^{\pi^2} \cdot 1 = e^{\pi^2}.$$

(b) Evaluate $f(x, y) = x^2 y - x + 3y^2 + 4$ at $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$.
We can evaluate the function by plugging in the x and y values into $f(x, y)$ as follows. First, we find that $f(0, 0)$ is

$$f(0, 0) = 0^2 \cdot 0 - 0 + 3 \cdot 0^2 + 4 = 4.$$

Then, we find that $f(1, 0)$ is

$$f(1, 0) = 1^2 \cdot 0 - 1 + 3 \cdot 0^2 + 4 = -1 + 4 = 3.$$

Then, we find that $f(0, 1)$ is

$$f(0, 1) = 0^2 \cdot 1 - 0 + 3 \cdot 1^2 + 4 = 3 + 4 = 7.$$

Finally, we find that $f(1, 1)$ is

$$f(1, 1) = 1^2 \cdot 1 - 1 + 3 \cdot 1^2 + 4 = 1 - 1 + 3 + 4 = 7.$$

(c) Evaluate $f(x, y) = \sqrt{x + y + \sin(\pi x y)}$ at $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$.
We can evaluate the function by plugging in the x and y values into $f(x, y)$ as follows. First, we find that $f(0, 0)$ is

$$f(0, 0) = \sqrt{0 + 0 + \sin(\pi \cdot 0 \cdot 0)} = \sqrt{0} = 0.$$

Then, we find that $f(1, 0)$ is

$$f(1, 0) = \sqrt{1 + 0 + \sin(\pi \cdot 1 \cdot 0)} = \sqrt{1 + 0 + 0} = 1.$$

Then, we find that $f(0,1)$ is

$$f(0,1) = \sqrt{0+1+\sin(\pi \cdot 0 \cdot 1)} = \sqrt{0+1+0} = 1.$$

Finally, we find that $f(1,1)$ is

$$f(1,1) = \sqrt{1+1+\sin(\pi \cdot 1 \cdot 1)} = \sqrt{2+\sin(\pi)} = \sqrt{2+0} = \sqrt{2}.$$

Problem 2. For each of the following functions $f(x,y)$, compute the two partial derivatives $\partial_x f$ and $\partial_y f$.

(a) $f(x,y) = e^{xy}\cos(x+y)$.
We compute the partial derivative $\partial_x f$ by treating y as a constant. Thus, we find that, using the product rule,

$$\partial_x f = e^{xy} y(\cos(x+y)) + -\sin(x+y)e^{xy} = e^{xy}\left(y\cos(x+y) - \sin(x+y)\right).$$

Similarly, to compute the partial derivative $\partial_y f$, we treat x as a constant. Hence, via the product rule, we get

$$\partial_y f = e^{xy} x\cos(x+y) + -\sin(x+y)e^{xy} = e^{xy}\left(x\cos(x+y) - \sin(x+y)\right).$$

(b) $f(x,y) = x^2 y - x + 3y^2 + 4$.
We compute the partial derivative $\partial_x f$ by treating y as a constant. Thus, we find that

$$\partial_x f = 2xy - 1 + 0 = 2xy - 1.$$

Similarly, to compute the partial derivative $\partial_y f$, we treat x as a constant. Hence, we get

$$\partial_y f = x^2 - 0 + 6y = x^2 - 6y.$$

(c) $f(x,y) = \sqrt{x+y+\sin(\pi xy)}$.
We compute the partial derivative $\partial_x f$ by treating y as a constant. Thus, using the chain rule, we find that

$$\partial_x f = \frac{1}{2}\left(x+y+\sin(\pi xy)\right)^{-1/2}\left(1+0+\cos(\pi xy)(\pi y)\right) = \frac{1+\pi y\cos(\pi xy)}{2\sqrt{x+y+\sin(\pi xy)}}.$$

Similarly, to compute the partial derivative $\partial_y f$, we treat x as a constant. Hence, via the chain rule, we get

$$\partial_y f = \frac{1}{2}\left(x+y+\sin(\pi xy)\right)^{-1/2}\left(0+1+\cos(\pi xy)(\pi x)\right) = \frac{1+\pi x\cos(\pi xy)}{2\sqrt{x+y+\sin(\pi xy)}}.$$

(d) $f(x,y) = x^7 \ln(y^3\cos(x^2)) - e^{xy^2}\sqrt{y}$.
We compute the partial derivative $\partial_x f$ by treating y as a constant. Thus, using

the chain rule and the product rule, we find that

$$\begin{aligned}
\partial_x f &= 7x^6 \ln(y^3 \cos(x^2)) + x^7 \frac{1}{y^3 \cos(x^2)} (y^3 \cdot - \sin(x^2) \cdot 2x) - e^{xy^2} \cdot y^2 \cdot \sqrt{y} \\
&= 7x^6 \ln(y^3 \cos(x^2)) + \frac{x^7 \cdot (-2xy^3 \sin(x^2))}{y^3 \cos(x^2)} - e^{xy^2} y^{5/2} \\
&= 7x^6 \ln(y^3 \cos(x^2)) + \frac{2x^8 y^3 \sin(x^2)}{y^3 \cos(x^2)} - e^{xy^2} y^{5/2} \\
&= 7x^6 \ln(y^3 \cos(x^2)) + 2x^8 \tan(x^2) - e^{xy^2} y^{5/2}.
\end{aligned}$$

Similarly, to compute the partial derivative $\partial_y f$, we treat x as a constant. Hence, via the chain rule and the product rule, we get

$$\begin{aligned}
\partial_y f &= x^7 \frac{1}{y^3 \cos(x^2)} (3y^2 \cos(x^2)) - e^{xy^2} \cdot x \cdot 2y \cdot \sqrt{y} - e^{xy^2} \cdot \frac{1}{2\sqrt{y}} \\
&= \frac{3y^2 x^7}{y^3} - e^{xy^2} \cdot 2xy^{3/2} - e^{xy^2} \cdot \frac{1}{2\sqrt{y}} \\
&= \frac{3x^7}{y} - e^{xy^2} \left(2xy^{3/2} + \frac{1}{2\sqrt{y}} \right).
\end{aligned}$$

Problem 3. For each of the following functions $f(x, y)$, evaluate the partial derivatives $\partial_x f$ at the indicated points.

(a) For $f(x, y) = e^{xy} \cos(x + y)$, evaluate $\partial_x f$ at $(x, y) = (0, 0)$.
We found $\partial_x f$ in the previous problem, which was

$$\partial_x f = e^{xy} \left(y \cos(x + y) - \sin(x + y) \right).$$

Then, we can evaluate this at $(0, 0)$ and we find that

$$\partial_x f\big|_{(x,y)=(0,0)} = e^{0 \cdot 0} \left(0 \cos(0 + 0) - \sin(0 + 0) \right) = 1 \cdot 0 = 0.$$

(b) For $f(x, y) = x^2 y - x + 3y^2 + 4$, evaluate $\partial_x f$ at $(x, y) = (5, 1)$.
We found $\partial_x f$ in the previous problem, which was

$$\partial_x f = 2xy - 1.$$

Then, we can evaluate this at $(5, 1)$ and we find that

$$\partial_x f\big|_{(x,y)=(5,1)} = 2 \cdot 5 \cdot 1 - 1 = 10 - 1 = 9.$$

(c) For $f(x, y) = \sqrt{x + y + \sin(\pi xy)}$, evaluate $\partial_x f$ at $(x, y) = (2, 0)$.
We found $\partial_x f$ in the previous problem, which was

$$\partial_x f = \frac{1 + \pi y \cos(\pi xy)}{2\sqrt{x + y + \sin(\pi xy)}}.$$

Then, we can evaluate this at $(2, 0)$ and we find that

$$\partial_x f\big|_{(x,y)=(2,0)} = \frac{1 + \pi \cdot 0 \cos(\pi \cdot 2 \cdot 0)}{2\sqrt{2 + 0 + \sin(\pi \cdot 2 \cdot 0)}} = \frac{1 + 0}{2\sqrt{2 + 0 + 0}} = \frac{1}{\sqrt{2}}.$$

(d) For $f(x,y) = x^7\ln(y^3\cos(x^2)) - e^{xy^2}\sqrt{y}$, evaluate $\partial_x f$ at $(x,y) = (\pi, 1)$.
We found $\partial_x f$ in the previous problem, which was

$$\partial_x f = 7x^6\ln(y^3\cos(x^2)) + 2x^8\tan(x^2) - e^{xy^2}y^{5/2}.$$

Then, we can evaluate this at $(\pi, 1)$ and we find that

$$\begin{aligned} \partial_x f\Big|_{(x,y)=(\pi,1)} &= 7(\pi)^6\ln(1^3\cos((\pi)^2)) + 2(\pi)^8\tan((\pi)^2) - e^{\pi\cdot 1^2}1^{5/2} \\ &= 7\pi^6\ln(\cos(\pi^2)) + 2\pi^8\tan(\pi^2) - e^{\pi}. \end{aligned}$$

Problem 4. For each of the following functions $f(x,y)$, show that $(x,y) = (0,0)$ is the only critical point:

(a) $f(x,y) = x^2 + y^2$.
To find a critical point of a function, we have to show that $\partial_x f$ and $\partial_y f$ are both 0 at this critical point. Thus, we first need to compute both of the partial derivatives. The partial derivative with respect to x is

$$\partial_x f = 2x$$

and the partial derivative with respect to y is

$$\partial_y f = 2y.$$

We see that

$$\partial_x f = 0 \text{ if and only if } x = 0$$

and

$$\partial_y f = 0 \text{ if and only if } y = 0.$$

Thus, the only critical point is $(x,y) = (0,0)$.

(b) $f(x,y) = x^2 - y^2$.
Similarly to the last question, need to first compute both of the partial derivatives. The partial derivative with respect to x is

$$\partial_x f = 2x$$

and the partial derivative with respect to y is

$$\partial_y f = -2y.$$

We see that

$$\partial_x f = 0 \text{ if and only if } x = 0$$

and

$$\partial_y f = 0 \text{ if and only if } y = 0.$$

Thus, the only critical point is $(x,y) = (0,0)$.

(c) $f(x,y) = -x^2 - y^2$.
Similarly to the previous questions, need to first compute both of the partial derivatives. The partial derivative with respect to x is

$$\partial_x f = -2x$$

and the partial derivative with respect to y is

$$\partial_y f = -2y.$$

We see that

$$\partial_x f = 0 \text{ if and only if } x = 0$$

and

$$\partial_y f = 0 \text{ if and only if } y = 0.$$

Thus, the only critical point is $(x, y) = (0, 0)$.

Problem 5. For each of the following functions $f(x, y)$, find *all* the critical points:

(a) $f(x, y) = (x - 5)^2 - (y + 7)^2 + 5$.
To find a critical point of a function, we have to show that $\partial_x f$ and $\partial_y f$ are both 0 at the critical points. Thus, we first need to compute both of the partial derivatives. The partial derivative with respect to x is

$$\partial_x f = 2(x - 5)$$

and the partial derivative with respect to y is

$$\partial_y f = 2(y - 7).$$

We see that

$$\partial_x f = 0 \text{ if and only if } x = 5$$

and

$$\partial_y f = 0 \text{ if and only if } y = 7.$$

Thus, the only critical point is $(x, y) = (5, 7)$.

(b) $f(x, y) = e^{-x^2 - y^2}$.
To find a critical point of a function, we have to show that $\partial_x f$ and $\partial_y f$ are both 0 at the critical points. Thus, we first need to compute both of the partial derivatives. The partial derivative with respect to x is

$$\partial_x f = e^{-x^2 - y^2} \cdot -2x$$

and the partial derivative with respect to y is

$$\partial_y f = e^{-x^2 - y^2} \cdot -2y.$$

Since exponentials will never be 0 for finite values of x and y, we then see that

$$\partial_x f = 0 \text{ if and only if } x = 0$$

and

$$\partial_y f = 0 \text{ if and only if } y = 0.$$

Thus, the only critical point is $(x, y) = (0, 0)$.

(c) $f(x, y) = 2x^3 + 6xy^2 - 3y^3 - 150x$.
To find a critical point of a function, we have to show that $\partial_x f$ and $\partial_y f$ are both 0 at the critical points. Thus, we first need to compute both of the partial derivatives. The partial derivative with respect to x is

$$\partial_x f = 6x^2 + 6y^2 - 150$$

and the partial derivative with respect to y is

$$\partial_y f = 12xy - 9y^2.$$

We can then set both of these equal to 0. First we consider that

$$\partial_y f = 12xy - 9y^2 = 0 \text{ if and only if } y(12x - 9y) = 0,$$

so then we find that the partial derivative with respect to y is zero only if

$$y = 0 \text{ or } y = 4/3x.$$

We then consider that

$$\partial_x f = 6x^2 + 6y^2 - 150 = 0 \text{ if and only if } x^2 + y^2 = 25.$$

We then consider the two cases from the ∂_y case and plug them in:
So if $y = 0$, then $x^2 = 25$, so $x = \pm 5$.
If $y = 4/3x$, then

$$\begin{aligned} x^2 + (4/3x)^2 &= 25 \\ (1 + 16/9)x^2 &= 25 \\ 25/9x^2 &= 25 \\ x^2 &= 9 \\ x &= \pm 3. \end{aligned}$$

Then, we can find that since $y = 4/3x$, then $y = (4/3)(\pm 3) = \pm 4$.
From this we can conclude we have 4 critical points. If $y = 0$, then $x = \pm 5$, and if $x = \pm 3$, then $y = \pm 4$. Thus, we get the following critical points:

$$(x, y) = (5, 0), (-5, 0), (3, 4), (-3, -4).$$

(d) $f(x, y) = x^4 + y^4 - 4xy$.
To find a critical point of a function, we have to show that $\partial_x f$ and $\partial_y f$ are both 0 at the critical points. Thus, we first need to compute both of the partial derivatives. The partial derivative with respect to x is

$$\partial_x f = 4x^3 - 4y$$

and the partial derivative with respect to y is

$$\partial_y f = 4y^3 - 4x.$$

We can then set both of these equal to 0. First we consider that

$$\partial_x f = 4x^3 - 4y = 0 \text{ if and only if } x^3 = y,$$

so then we find that the partial derivative with respect to x is zero only if

$$y = x^3.$$

We then consider that

$$\partial_y f = 4y^3 - 4x = 0 \text{ if and only if } y^3 = x.$$

We then consider the results from the $\partial_x f$ case and we know that since $y = x^3$, then

$$x^9 = x.$$

The only three cases where $x^9 = x$ is when $x = 0, \pm 1$. Thus, we find that if $x = 0$, then $y = 0^3 = 0$. If $x = 1$, then $y = 1^3 = 1$, and if $x = -1$, then $y = (-1)^3 = -1$. Thus, we have the following three critical points:

$$(x, y) = (0, 0), (1, 1), (-1, -1).$$

(e) $f(x, y) = e^{x-y^2-x^3/3}$.
To find a critical point of a function, we have to show that $\partial_x f$ and $\partial_y f$ are both 0 at the critical points. Thus, we first need to compute both of the partial derivatives. The partial derivative with respect to x is

$$\partial_x f = e^{x-y^2-x^3/3}(1 - x^2)$$

and the partial derivative with respect to y is

$$\partial_y f = e^{x-y^2-x^3/3}(-2y).$$

We can then set both of these equal to 0. Since exponentials will never be 0 for finite values of x and y, we then see that we can divide out the exponential. First we consider that

$$\partial_x f = e^{x-y^2-x^3/3}(1 - x^2) = 0 \text{ if and only if } 1 - x^2 = 0,$$

so then we find that the partial derivative with respect to x is zero only if

$$x^2 = 1 \text{ or } x = \pm 1.$$

We then consider that

$$\partial_y f = e^{x-y^2-x^3/3}(-2y) = 0 \text{ if and only if } -2y = 0.$$

Thus, we see that $y = 0$. We can combine these results to find that we have the following two critical points:

$$(x, y) = (1, 0), (-1, 0).$$

(f) $f(x, y) = x + 3y - 17$.
To find a critical point of a function, we have to show that $\partial_x f$ and $\partial_y f$ are both 0 at the critical points. Thus, we first need to compute both of the partial derivatives. The partial derivative with respect to x is

$$\partial_x f = 1$$

and the partial derivative with respect to y is

$$\partial_y f = 3.$$

We can see that these partial derivatives can never be 0 anywhere, so there are no critical points.

Hint: The first two functions in (a), (b) have each only one critical point. The one in (c) has four critical points, the one in (d) has three and the one in (e) has two critical points. Finally, the function in (f) has none.