MAT 21C: PRACTICE PROBLEMS LECTURE 12

PROFESSOR CASALS (SECTIONS B01-08)

ABSTRACT. Practice problems for the twelfth lecture of Part II, delivered June 2nd 2023. Solutions will be posted within 48h of these problems being posted.

Brief reminder from lecture: Given a critical point (x_0, y_0) of a function $f(x, y)$, the characteristic polynomial $p(\lambda)$ at the critical point (x_0, y_0) is computed as follows:

(1) First, compute the four second derivatives
$$\partial_{xx} f, \partial_{yy} f, \partial_{xy} f, \partial_{yx} f.$$
(Recall that you should always get $\partial_{xy} f = \partial_{yx} f$.)

(2) Second, evaluate the four second derivatives at (x_0, y_0), setting:
$$a := \partial_{xx} f(x_0, y_0), \quad b := \partial_{xy} f(x_0, y_0),$$
$$c := \partial_{yx} f(x_0, y_0), \quad d := \partial_{yy} f(x_0, y_0).$$

(3) The characteristic polynomial $p(\lambda)$ at the critical point (x_0, y_0) is the polynomial
$$\lambda^2 - (a + d)\lambda + (ad - bc).$$

The real roots λ_+, λ_- of the characteristic polynomial determine the type of critical point, as follows:

- If both $\lambda_+, \lambda_- > 0$, then it is a minimum.
- If both $\lambda_+, \lambda_- < 0$, then it is a maximum.
- If one of λ_+, λ_- is positive and the other is negative, then it is a saddle.
- Otherwise, i.e. if at least one of λ_+, λ_- is zero or the roots are imaginary, then we cannot decide.

Another fast way to check is to just compute a and $ad - bc$ directly, then you can instead use the following criteria:

- If both $ad - bc > 0$ and $a > 0$, then it is a minimum.
- If $ad - bc > 0$ and $a < 0$, then it is a maximum.
- If $ad - bc < 0$, then it is a saddle.
- If $ad - bc = 0$, then we cannot decide.

Problem 1. Consider the function $f(x,y) = x^2y^2 - 5x^2 - 5y^2 - 8xy$.

(a) Show that the critical points are $(0,0)$, $(3,3)$, $(-3,-3)$, $(1,-1)$ and $(-1,1)$.

The first derivatives are $f_x(x,y) = 2xy^2 - 10x - 8y$ and $f_y(x,y) = 2x^2y - 10y - 8x$. Since $f_x(0,0) = f_y(0,0) = 0, f_x(3,3) = f_y(3,3) = 0, f_x(-3,-3) = f_y(-3,-3) = 0, f_x(1,-1) = f_y(1,-1) = 0$, and $f_x(-1,1) = f_y(-1,1) = 0$, those points are critical points.

(b) Compute all the second derivatives $\partial_{xx}f, \partial_{yy}f, \partial_{xy}f, \partial_{yx}f$.

$\partial_{xx}f = 2y^2 - 10$
$\partial_{yy}f = 2x^2 - 10$
$\partial_{xy}f = 4xy - 8$
$\partial_{yx}f = 4xy - 8$

(c) Write the characteristic polynomials for each of the 5 critical points in Part (a).

At $(0,0)$: $\lambda^2 + 20\lambda + 100$
At $(3,3)$: $\lambda^2 - 16\lambda + (16 - 28^2)$
At $(-3,-3)$: $\lambda^2 - 16\lambda + (16 - 28^2)$
At $(1,-1)$: $\lambda^2 + 16\lambda + (-16 - (-12)^2)$
At $(-1,1)$: $\lambda^2 + 16\lambda + (-16 - (-12)^2)$

(d) Show that $(0,0)$ is a maximum, and all the rest, $(3,3)$, $(-3,-3)$, $(1,-1)$ and $(-1,1)$, are saddle points.

For (0,0), $\lambda^2 + 20\lambda + 100 = (\lambda + 10)^2 = 0 \implies \lambda_+, \lambda_- = -10$, so there is a maximum at this point. We see that $16 - 28^2 < 0$ and $-16 - (-12)^2 < 0$, so the rest are saddle points (by the shortcut).

Problem 2. Consider the function $f(x,y) = 4x^2 + 9y^2 + 8x - 36y + 24$.

(a) Show that the only critical point is $(-1,2)$.

$f_x(x,y) = 8x + 8 = 0 \implies x = 1$
$f_y(x,y) = 18y - 36 = 0 \implies y = 2$

(b) Prove that $(-1,2)$ is a minimum.

$\partial_{xx}f = 8$

$\partial_{yy}f = 18$

$\partial_{xy}f = \partial_{yx}f = 0$

Since $8(18) - 0 > 0$ and $8 > 0$, the point (-1,2) is a minimum

Problem 3. For each of the following functions, find *all* critical points and determine whether they are minima, saddles, maxima or cannot decide.

(a) $f(x,y) = x^3 + 2xy - 6x - 4y^2$.
$(x_0, y_0) = (-\frac{3}{2}, -\frac{3}{8})$ - maximum
$(x_1, y_1) = (\frac{4}{3}, \frac{1}{3})$ - saddle

(b) $f(x,y) = x^3 - 3xy^2$.
$(x_0, y_0) = (0,0)$ - saddle

(c) $f(x,y) = e^x(x^4 + y^4)$.
$(x_0, y_0) = (-4, 0)$ - can't decide
$(x_1, y_1) = (0, 0)$ - can't decide

(d) $f(x,y) = xy - x + y$.
$(x_0, y_0) = (-1, 1)$ - saddle

(e) $f(x,y) = y\cos(x)$.
$(x_n, y_n) = (2\pi n \pm \frac{\pi}{2}, 0)$ for $n \in \mathbb{N}$ - (infinitely many) saddle points

(f) $f(x,y) = x^2y^2$.
$(x_0, y_0) = (0, 0)$ - can't decide