MAT 21C: SOLUTIONS TO PROBLEM SET 1

Problem 1. From "Exercises 10.1" in textbook, in the "Convergence and Divergence" solve 31 through 42 (each item is worth 2 points) and 43 (worth 1 point).

Solutions to the odd numbered exercises can be found in the textbook.

Problem 2. "Exercises 10.1" in textbook in "Convergence and Divergence", solve:

- 101 through 105 (each of these items is worth 2 points)
- 121 through 127 (each of these items is worth 2 points)
- 134 (worth 1 point).

Solutions to the odd numbered exercises can be found in the textbook.

Problem 3. For each of the following statements, justify whether they are true or explain why they are false (e.g. providing a counter-example):

(a) A divergent sequence (a_n) must be unbounded.

False. The sequence $a_n = (-1)^n$ is bounded and divergent.

(b) A convergent sequence (a_n) must be either decreasing or increasing.

False. The sequence $a_n = \dfrac{(-1)^n}{n}$ is convergent but not decreasing or increasing. (It is not even eventually increasing or decreasing).

(c) If (a_n) and (b_n) are convergent, then the quotient sequence whose nth term is the quotient $\dfrac{a_n}{b_n}$ is also convergent.

False. The sequences $a_n = 1$ and $b_n = \dfrac{1}{n}$ are convergent but $\dfrac{a_n}{b_n} = n$ is divergent.

(d) Let (a_n) and (c_n) be convergent sequences. Then any sequence (b_n) such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$, i.e. (b_n) lies between (a_n) and (c_n), is also convergent.

False. The sequences $a_n = -2$ and $b_n = 2$ are convergent but $c_n = (-1)^n$ is divergent, even though it lies between (a_n) and (b_n).

(e) A sequence (a_n) whose evenly-indexed terms a_{2n} are positive and whose odd-indexed terms a_{2n+1} are negative cannot be convergent.

False. The sequence $a_n = \dfrac{(-1)^n}{n}$ converges and satisfies the conditions.

(*Each item is worth 5 points.*)

Problem 4. For each of the following statements, circle all the correct answers. It might be that no answers are correct, some answers are correct or all answers are correct. If no answers are correct please directly write "No answer is correct.", otherwise circle accordingly.

Just in case: If an arbitrary sequence (a_n) is given, statements such as "(a_n) is increasing" means "(a_n) is increasing for any such given (a_n)". So if an answer is sometimes correct and sometimes incorrect, depending on the choice of possible (a_n), then it must **not** *be circled.*

Solutions marked in **bolded font**.

(1) The sequence $a_n = \dfrac{3^n}{n!}$, $n \geq 1$, is

(1) increasing. (2) decreasing. **(3) convergent**. **(4) bounded**.

(2) If (a_n) is a divergent sequence of positive real numbers and (b_n) is such that $a_n \leq b_n$ for all $n \in \mathbb{N}$, then

(1) (b_n) diverges. (2) (b_n) is unbounded. (3) (b_n) converges.

No answer is correct.

(3) Let (a_n) be a sequence of *rational* numbers. Then

(1) (a_n) is divergent. (2) If (a_n) converges the limit is rational too. (3) If (a_n) diverges, it is unbounded. (4) (a_n) must be bounded.

No answer is correct.

(4) The sequence $a_n = n^2 \ln(n)$ growth slower than

(1) $\ln(n)$. (2) n^2. **(3)** n^3. (4) $n \ln(n)$. (5) $n \ln(\ln(n))$.

(5) The sequence $a_n = (\sqrt{2})^{a_{n-1}}$, $a_1 = \sqrt{2}$, defined recursively, is:

(1) Increasing. **(2) Bounded.** **(3) Convergent.** (4) Unbounded.

(6) Suppose (a_n) converges and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. Then the sequence $f(a_n)$

(1) converges. **(2) is bounded.** (3) is increasing. (4) is decreasing.

(7) Suppose (a_n) diverges, then the sequence $\cos(a_n)$

(1) diverges. (2) is unbounded. **(3) might converge or diverge.**[1]

(8) Suppose that (a_n) diverges. Then the sequence (b_n) given by $b_n = a_n^2$

[1] Depending on the given sequence (a_n).

(1) diverges. (2) converges. (3) is bounded. (4) is increasing.

No answer is correct.

(9) Suppose (a_n) is a sequence, $f : \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function and the sequence $f(a_n)$ converges. Then the sequence (a_n)

(1) converges. (2) is bounded. (3) increases. **(4) has bounded image $f(a_n)$.**

(10) Let (a_n) be defined as $a_n =$ "sum of the first n odd natural numbers". Then the sequence (b_n) given by $b_n := \dfrac{a_n}{n^3 \ln(n)}$, defined for $n \geq 2$,

(1) is bounded. **(2) converges.** (3) is increasing.

(*The first 5 items are worth 3 points, the last 5 items are worth 2 points.*)