MAT 21C: PROBLEM SET 2

DUE TO FRIDAY APR 21 2023

Abstract. This problem set corresponds to the first week of the course MAT-21C Spring 2023. It is due Friday Apr 21 at 9:00am submitted via Gradescope.

Purpose: The goal of this assignment is to acquire the necessary skills to work with infinite series. These were discussed during the second week of the course and are covered in Sections 10.2, 10.3, 10.4, 10.5 and 10.6 of the textbook.

Task: Solve Problems 1 through 4 below.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

You are welcome to use the **Calculus Room** at the Department of Mathematics, and the Office Hours offered by the Professor and the Recitation Instructors. Again, list any collaborators or contributors in your solutions. Make sure you are using your own thought process and words, even if an idea or solution came from elsewhere. (In particular, it might be wrong, so please make sure to think about it yourself.)

Grade: Each graded Problem is worth 25 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Thomas' Calculus Early Transcendentals" (15th Edition) by G. B. Thomas, , and J. Hass, Christopher E. Heil, P. Bogacki and M. Weir.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct. *In general, you must give a precise reason for why a sequence is divergent or convergent. Similarly, if the sequence is converge, please justify to the best of your ability the claimed limit in your answer. If you are using theorems in lecture and in the textbook, make that reference clear. (E.g. specify name/number of the theorem and section of the book.)*

Problem 1. From "Exercises 10.2" in textbook, solve **39, 41, 44, 45, 50, 53, 58, 81, 85, 86** (each item is worth 2 points) and From "Exercises 10.3" in textbook, solve **5, 6, 8, 21, 29** (each item is worth 1 points).

Problem 2. From the textbook, solve:

- In "Exercises 10.4", **11, 12, 19, 26, 27** (each item is worth 2 points)
- In "Exercises 10.5", **1,4,6,9,11,14** (each of these items is worth 2 points)
- In "Exercises 10.6", **2, 27, 35** (worth 1 point).

Problem 3. For each statement, justify whether they are true or explain why they are false (providing a counter-example). Each item is worth 5 points.

(a) If $(a_n) \to 0$ then the series $\sum_{n=1}^{\infty} a_n$ converges.

(b) If the root test is inconclusive, then the ratio test is inconclusive.

(c) Let (a_n) be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous positive decreasins function of x for all $x \geq 1$. If the series $\sum_{n=1}^{\infty} a_n$ converges then we have the equality

$$\sum_{n=1}^{\infty} a_n = \int_1^{\infty} f(x)dx.$$

(d) If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} |a_n|$ converges.

(e) There are series $\sum_{n=1}^{\infty} a_n$ for which the integral test determines convergence but the root test does not.

Problem 4. In this problem we study the convergence of the series $S := \sum_{n=1}^{\infty} e^{-n^2}$ from the perspective of the different tests. Each item is worth 5 points.

(i) Show that $\sum_{n=1}^{\infty} e^{-n}$ converges and its limit[1] is

$$\sum_{n=1}^{\infty} e^{-n} = \frac{1}{e-1}.$$

(ii) Deduce from (i) that S is convergent by comparing it to $\sum_{n=1}^{\infty} e^{-n}$.

(iii) Use the integral test to show that S converges.

Hint: You may use the beautiful equality $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

(iv) Use the ratio test to show that S converges.

(v) Use the root test to show that S converges.

For the record, the exact value of $\sum_{n=1}^{\infty} e^{-n^2}$ is actually $(1 + \vartheta_3(0, e^{-1}))/2$, where ϑ_3 is a Jacobi theta function, which encodes things such as heat dispersion, the translational partition function for an ideal gas or how natural numbers can be expressed as sums of (four) squares.

[1] There was a typo before, where it said that the geometric series converged to $e/(e-1)$. It should be $1/(e-1)$ because the series as written starts at $n = 1$.