MAT 21C: PRACTICE PROBLEMS LECTURE 12

PROFESSOR CASALS (SECTIONS B01-08)

Abstract. Practice problems for the twelfth lecture of Part II, delivered June 2nd 2023. Solutions will be posted within 48h of these problems being posted.

Brief reminder from lecture: Given a critical point (x_0, y_0) of a function $f(x, y)$, the characteristic polynomial $p(\lambda)$ at the critical point (x_0, y_0) is computed as follows:

(1) First, compute the four second derivatives
$$\partial_{xx} f, \partial_{yy} f, \partial_{xy} f, \partial_{yx} f.$$
(Recall that you should always get $\partial_{xy} f = \partial_{yx} f$.)

(2) Second, evaluate the four second derivatives at (x_0, y_0), setting:
$$a := \partial_{xx} f(x_0, y_0), \quad b := \partial_{xy} f(x_0, y_0),$$
$$c := \partial_{yx} f(x_0, y_0), \quad d := \partial_{yy} f(x_0, y_0).$$

(3) The characteristic polynomial $p(\lambda)$ at the critical point (x_0, y_0) is the polynomial
$$\lambda^2 - (a + d)\lambda + (ad - bc).$$

The real roots λ_+, λ_- of the characteristic polynomial determine the type of critical point, as follows:

- If both $\lambda_+, \lambda_- > 0$, then it is a minimum.
- If both $\lambda_+, \lambda_- < 0$, then it is a maximum.
- If one of λ_+, λ_- is positive and the other is negative, then it is a saddle.
- Otherwise, i.e. if at least one of λ_+, λ_- is zero or the roots are imaginary, then we cannot decide.

Another fast way to check is to just compute a and $ad - bc$ directly, then you can instead use the following criteria:

- If both $ad - bc > 0$ and $a > 0$, then it is a minimum.
- If $ad - bc > 0$ and $a < 0$, then it is a maximum.
- If $ad - bc < 0$, then it is a saddle.
- If $ad - bc = 0$, then we cannot decide.

Problem 1. Consider the function $f(x,y) = x^2y^2 - 5x^2 - 5y^2 - 8xy$.

(a) Show that the critical points are $(0,0)$, $(3,3)$, $(-3,-3)$, $(1,-1)$ and $(-1,1)$.

(b) Compute all the second derivatives $\partial_{xx}f, \partial_{yy}f, \partial_{xy}f, \partial_{yx}f$.

(c) Write the characteristic polynomials for each of the 5 critical points in Part (a).

(d) Show that $(0,0)$ is a maximum, and all the rest, $(3,3)$, $(-3,-3)$, $(1,-1)$ and $(-1,1)$, are saddle points.

Problem 2. Consider the function $f(x,y) = 4x^2 + 9y^2 + 8x - 36y + 24$.

(a) Show that the only critical point is $(-1,2)$.

(b) Prove that $(-1,2)$ is a minimum.

Problem 3. For each of the following functions, find *all* critical points and determine whether they are minima, saddles, maxima or cannot decide.

(a) $f(x,y) = x^3 + 2xy - 6x - 4y^2$.

(b) $f(x,y) = x^3 - 3xy^2$.

(c) $f(x,y) = e^x(x^4 + y^4)$.

(d) $f(x,y) = xy - x + y$.

(e) $f(x,y) = y\cos(x)$.

(f) $f(x,y) = x^2y^2$.