MAT 21C: PRACTICE PROBLEMS LECTURE 3

PROFESSOR CASALS (SECTIONS B01-08)

Abstract. Practice problems for the third lecture of Part II, delivered May 5 2023.

Problem 1. Show that the surface $S = \{x^2 - 2x + y^2 + z^2 - 6z = -1\}$ is the sphere of radius 3 and center $(1, 0, 3)$.

We want to verify that the equation that gives S is equivalent to the equation

$$(x-1)^2 + y^2 + (z-3)^2 = 9,$$

which indeed describes the sphere of radius 3 and center $(1, 0, 3)$. For that we consider the above equation and expand it:

$$(x-1)^2 + y^2 + (z-3)^2 = 9 \iff x^2 - 2x + 1 + y^2 + z^2 - 6z + 9 = 9 \iff$$
$$\iff x^2 - 2x + 1 + y^2 + z^2 - 6z = 0 \iff x^2 - 2x + 1 + y^2 + z^2 - 6z = -1$$

which is indeed the equation given in the statement of the problem. Here we have used that $(a-b)^2 = a^2 - 2ab + b^2$.

Problem 2. Knowing that the surface $S = \{x^2 + y^2 + 8y + z^2 - 2z = 8\}$ is a sphere of radius 5, find its center.

Since $x^2 + y^2 + 8y + z^2 - 2z = 8$ is a sphere of radius 5, its equation must be of the form

$$(x-c_1)^2 + (y-c_2)^2 + (z-c_3)^2 = 25,$$

where (c_1, c_2, c_3) is its center. We need to find the (unique) values of (c_1, c_2, c_3) so that this equation matches the equation $x^2 + y^2 + 8y + z^2 - 2z = 8$. Expanding the equation above we obtain:

$$x^2 - 2xc_1 + c_1^2 + y^2 - 2yc_2 + c_2^2 + z^2 - 2zc_3 + c_3^2 = 25$$

We want to be comparing the coefficients in front of each x^2, y^2, z^2, x, y, z and the constant for each of this equation and $x^2 + y^2 + 8y + z^2 - 2z = 8$:

- For x^2, y^2, z^2 the coefficients are 1 for both, so that is a good start.
- For x the coefficients are 0 and $-2c_1$, so we obtain the equation $0 = -2c_1$. Therefore $c_1 = 0$.
- For y the coefficients are 8 and $-2c_2$, so we obtain the equation $8 = -2c_2$. Therefore $c_2 = -4$.
- For z the coefficients are -2 and $-2c_3$, so we obtain the equation $-2 = -2c_3$. Therefore $c_3 = 1$.
- The constant coefficient are $25 - c_1^2 - c_2^2 - c_3^2$ and 8. Since $25 - 16 - 1 = 8$, the above values of $c_1 = 0, c_2 = -4$ and $c_3 = -1$ work.

In conclusion, the center of the sphere is $(c_1, c_2, c_3) = (0, -4, 1)$.

Problem 3. Describe the set of points satisfying

$$\{x + 2y - z = 3, \quad 4x - 5y - z = 0\}.$$

The set of points satisfying $x + 2y - z = 3$ is a plane. The set of points satisfying $4x - 5y - z = 0$ is also a plane. Therefore the set of points satisfying both equations is the intersection of these two planes.

Two planes can either *not* intersect (at all) or intersect exactly at a line. If we find a point that belongs to both planes, it must be that they intersect and therefore the intersection will have to be a line. So we need to find one solution of both equations $4x - 5y - z = 0$ and $x + 2y - z = 3$. Subtracting one equation from the other we deduce that $3x - 7y = -3$ and we can just choose $x = -1$ and $y = 0$ as a possible solution. Since $x + 2y - z = 3$ (or $4x - 5y - z = 0$), we get $z = -4$. Hence the point $(-1, 0, -4)$ belongs to both planes.

Problem 4. Let $\pi_1 = \{ax + by + cz = d\}$ and $\pi_2 = \{\alpha x + \beta y + \gamma z = \delta\}$ be two planes. Show that π_1 are π_2 are parallel (and therefore do not intersect) if and only if (a, b, c) is a multiple of (α, β, γ), i.e. $a/\alpha = b/\beta = c/\gamma$.

The perpendicular direction to π_1 is (a, b, c) and the perpendicular direction to π_2 is (α, β, γ). The two planes π_1 and π_2 being parallel is equivalent to saying that their perpendicular directions are "in the same direction", i.e. one is a multiple of the other. Therefore π_1 and π_2 are parallel if and only if there exists a real constant k such that $(k \cdot a, k \cdot b, k \cdot c) = (\alpha, \beta, \gamma)$. In that case, the ratios $a/\alpha, b/\beta$ and c/γ are all equal to k and thus equal to each other.

Problem 5. Describe the following subsets of space:

(1) $\{x - y + 3z \geq 4\}$

(2) $\{x \geq 0, y \geq 0, z \geq 0\}$

(3) $\{x^2 - 4x + 4 + y^2 + z^2 < 81\}$

(4) $\{x^2 + y^2 + z^2 < 36, y \leq 0\}$

(5) $\{x^2 + y^2 + z^2 < 36, y = 0\}$

(6) $\{x^2 + y^2 + z^2 = 36, y = 0\}$

(7) $\{x^2 + y^2 + z^2 > 36, y = 0\}$

For (1), the equality $x - y + 3z = 4$ defines a plane in space. Hence it cuts space out into two half-spaces. Therefore $x - y + 3z \geq 4$ is a half-space. To decide which one, we note that $(0, 0, 0)$ satisfies$x - y + 3z < 4$. Therefore, the answer is that it is the only half-space (out of the possible two) which does *not* contain the origin.

For (2), $x = 0$ cuts out a plane and thus $x \geq 0$ is a half-space. It is the half-space that contains $(1, 0, 0)$, for instance. (Instead of, say, $(-1, 0, 0)$.) Note that a half-space is made out of 4 octants. Now $y = 0$ also cuts out a plane and $y \geq 0$ is also a half-space. The intersection of $x \geq 0$ with $y \geq 0$ is therefore the intersection of two half-spaces. This intersection is 2 of those 4 octants. Finally, $z \geq 0$ is also a half-space. The intersection of $z \geq 0$ with $\{x \geq 0, y \geq 0\}$ is 1 of octant (of the two in $\{x \geq 0, y \geq 0\}$).

For (3), we note that $\{x^2 - 4x + 4 + y^2 + z^2 = 81\}$ can be re-written as $\{(x-2)^2 + y^2 + z^2 = 9^2\}$, which we recognize as the sphere centered at $(2, 0, 0)$ of radius 9. This sphere cuts space into two halves: a ball, which contains the center $(2, 0, 0)$ and space minus a ball. To decide which of these two regions $\{x^2 - 4x + 4 + y^2 + z^2 < 81\}$ represents, we check what inequality the center verified. Since $(x, y, z) = (2, 0, 0)$ plug into $\{x^2 - 4x + 4 + y^2 + z^2 < 81\}$ gives $0 < 81$, the inequality is satisfied. Therefore $\{x^2 - 4x + 4 + y^2 + z^2 < 81\}$ is the region which contains the center and it is thus a ball.

For (4), $\{x^2 + y^2 + z^2 = 36\}$ is the sphere of radius $R = 6$ centered at the origin $O = (0, 0, 0)$. Therefore $\{x^2 + y^2 + z^2 < 36\}$ is a ball. The equality $y = 0$ cuts out a plane through the origin and $y \leq 0$ is a half-space. Therefore the intersection of the ball $\{x^2 + y^2 + z^2 < 36\}$ with the half-space $\{y \leq 0\}$ is half of the ball. (Note that the plane $\{y = 0\}$ contains the center, so it really cuts the ball exactly by half.)

For (5), $\{x^2 + y^2 + z^2 = 36\}$ is the sphere of radius $R = 6$ centered at the origin $O = (0, 0, 0)$ and, as before $\{x^2 + y^2 + z^2 < 36\}$ is a ball. The plane $\{y = 0\}$ passes through the center of this ball. Therefore the intersection of this ball with the plane is a disk: a flat surface contained in the plane $\{y = 0\}$ in the shape of a pizza.

For (6), $\{x^2 + y^2 + z^2 = 36\}$ is the sphere of radius $R = 6$ centered at the origin $O = (0, 0, 0)$. The plane $\{y = 0\}$ passes through the center of this ball. Now the intersection of this sphere (a surface, not a ball!) with the plane is a circle: it is a circle in the plane $\{y = 0\}$. In the previous example, that circle is the crust of the pizza disk in the previous example.

For (7), $\{x^2 + y^2 + z^2 = 36\}$ is the sphere of radius $R = 6$ centered at the origin $O = (0, 0, 0)$ and $\{x^2 + y^2 + z^2 > 36\}$ is the entire space minus a ball. (Like we removed a solid ball from space.) The equality $y = 0$ cuts out a plane through the origin. The intersection of the region $\{x^2 + y^2 + z^2 > 36\}$ with the plane $\{y = 0\}$ is contained in the plane. Since we are intersecting space minus a ball with the plane, and the ball intersects the plane at a disk, the answer is that the intersection is a plane minus a disk. (If you had a table with a pizza on it, this region is exactly all points in the table, which is the plane, *except* the regions that has the pizza.)