MAT 21C: PRACTICE PROBLEMS LECTURE 4

PROFESSOR CASALS (SECTIONS B01-08)

Abstract. Practice problems for the fourth lecture of Part II, delivered May 8 2023. Solutions will be posted within 48h of these problems being posted.

Problem 1. Consider the vectors $v = \langle 3, -7, 1\rangle$ and $w = \langle -1, 4, 2\rangle$.

(a) Compute the following vectors $v + w$, $2v$, $5w$ and $2v + 5w$.
For the sum of vectors v and w, we can add the vectors component wise as follows:

$$v + w = \langle 3, -7, 1\rangle + \langle -1, 4, 2\rangle = \langle 2, -3, 3\rangle.$$

For the product of a scalar and a vector, we multiply the scalar to each component of the vector. In this case, we multiply 2 to each component of v.

$$2v = 2\langle 3, -7, 1\rangle = \langle 6, -14, 2\rangle$$

Here, we multiply 5 to each component of w.

$$5w = 5\langle -1, 4, 2\rangle = \langle -5, 20, 10\rangle$$

Then, we can take the vectors $2v$ and $5w$, which were calculated in the previous step, and find their sum by adding the vectors component wise.

$$2v + 5w = \langle 6, -14, 2\rangle + \langle -5, 20, 10\rangle = \langle 1, 6, 12\rangle$$

(b) Find the length of v and w.
To calculate the length of any vector $u = \langle u_1, u_2, u_3\rangle$, we use the length or distance formula:

$$|u| = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

Thus, we can use this formula to calculate the length of v and w.

$$|v| = \sqrt{3^2 + (-7)^2 + 1^2} = \sqrt{9 + 41 + 1} = \sqrt{59}$$
$$|w| = \sqrt{(-1)^2 + 4^2 + 2^2} = \sqrt{1 + 16 + 4} = \sqrt{21}$$

(c) Find the unit length vector in the direction of v and the unit length vector in the direction of w.
To find the unit length vectors in the direction of v and w, we can divide the original vectors by their lengths. Ie, for v, we know v already points in the direction of v. So in order to find the unit length vector in the direction of v, we divide v by its length, which was calculated in the previous step. Then, we have a vector of length one, or unit length, pointing in the direction of v. We follow the same steps for w.

$$\frac{v}{|v|} = \left\langle \frac{3}{\sqrt{59}}, \frac{-7}{\sqrt{59}}, \frac{1}{\sqrt{59}} \right\rangle$$

$$\frac{w}{|w|} = \left\langle \frac{-1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, \frac{2}{\sqrt{21}} \right\rangle$$

Problem 2. Consider the points $P = (-5, 1, 2)$, $Q = (7, 6, -3)$, and $R = (1, 0, 4)$.

(a) Find the vector $\vec{PQ}$ and its length $|\vec{PQ}|$.
To find $\vec{PQ}$, we know this is the vector pointing from the point P to the point Q, so we take the vector from the origin to the point P, ie $\langle -5, 1, 2\rangle$, and subtract it from the vector from the origin to the point Q, ie $\langle 7, 6, -3\rangle$. Then we find that

$$\vec{PQ} = \langle 7, 6, -3\rangle - \langle -5, 1, 2\rangle = \langle 12, 5, -5\rangle.$$

Then, we can compute the length of that vector:

$$|\vec{PQ}| = \sqrt{12^2 + 5^2 + (-5)^2} = \sqrt{144 + 25 + 25} = \sqrt{194}$$

(b) Find the midpoint between P and Q.
To find the midpoint of the line segment between P and Q, we need to take the components of each of the coordinate points and average them. That goes as follows:

$$\left(\frac{-5+7}{2}, \frac{1+6}{2}, \frac{2-3}{2}\right) = \left(1, \frac{7}{2}, -\frac{1}{2}\right).$$

(c) Compute the vectors $\vec{QR}$ and $\vec{PR}$ and show that $\vec{PQ} + \vec{QR} = \vec{PR}$.
We can compute vectors $\vec{PR}$ and $\vec{QR}$ just as we computed $\vec{PQ}$ in part (a).

$$\vec{QR} = \langle 1, 0, 4\rangle - \langle 7, 6, -3\rangle = \langle -6, -6, 7\rangle$$

$$\vec{PR} = \langle 1, 0, 4\rangle - \langle -5, 1, 2\rangle = \langle 6, -1, 2\rangle$$

Then, we add $\vec{PQ} + \vec{QR}$ together component-wise, and we see that the sum is the same as $\vec{PR}$.

$$\vec{PQ} + \vec{QR} = \langle -6, -6, 7\rangle + \langle 12, 5, -5\rangle = \langle 6, -1, 2\rangle = \vec{PR}$$

Problem 3. Consider the vectors $v = \langle 1, -2, -3\rangle$ and $w = \langle -1, 4, 5\rangle$.

(a) Find a vector u such that $v + u = w$.
We can find a vector u is this manner by subtracting w from v component wise. We get as follows:

$$u = w - v = \langle -1, 4, 5\rangle - \langle 1, -2, -3\rangle = \langle -2, 6, 8\rangle.$$

(b) Compute the lengths $|v|, |w|$ and $|v + w|$.
We can compute the lengths of v and w as has been done previously in these practice problems.

$$|v| = \sqrt{1^2 + (-2)^2 + (-3)^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$|w| = \sqrt{(-1)^2 + 4^2 + 5^2} = \sqrt{1 + 16 + 25} = \sqrt{42}$$

For computing the length of v and w, note that we have to add the vectors together first before finding their length. Thus, we see that first

$$v + w = \langle 1, -2, -3\rangle + \langle -1, 4, 5\rangle = \langle 0, 2, 2\rangle.$$

Then we can compute their length:

$$|v+w| = \sqrt{(0)^2+(2)^2+(2)^2}\sqrt{0+4+4} = 2\sqrt{2}$$

(c) Justify geometrically that $|v+w| \leq |v|+|w|$.
We can see that the vectors form a triangle, and one side of a triangle is shorter than the sum of the lengths of the other two sides of the triangle. This inequality in the problem is called the triangle inequality. We can see this in Figure 1.

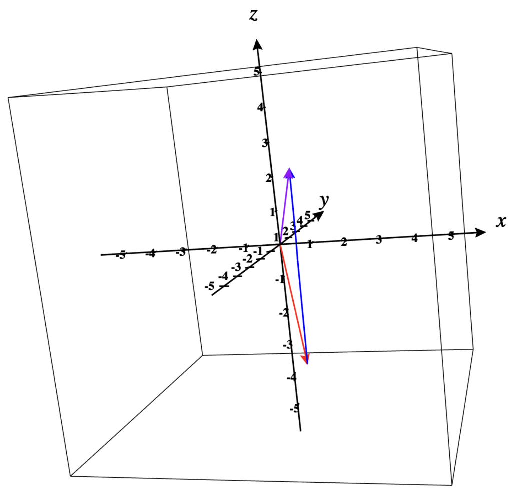

FIGURE 1. v is in red, w is in blue, and $v+w$ is in purple

Problem 4. Suppose that a vector $v = \langle v_1, v_2, v_3\rangle$ has length zero $|v| = 0$. Argue that then $v = \langle 0,0,0\rangle$, i.e. $v_1 = v_2 = v_3 = 0$.

Notice if $|v| = 0$, then

$$\sqrt{v_1^2+v_2^2+v_3^2} = 0,$$

by the definition of the length of a vector. If we square both sides of the equation, we find that

$$v_1^2+v_2^2+v_3^2 = 0.$$

Then, since $v_1^2, v_2^2, v_3^2 \geq 0$, since the squares of numbers are always positive or 0, but the sum of these positive or zero numbers is zero, then it must be true that

$$v_1^2 = v_2^2 = v_3^2 = 0.$$

Thus, if we take the square roots of all of these terms, we find that

$$v_1 = v_2 = v_3 = 0.$$

Problem 5. (Optional) Consider the eight vectors

$$v_1 = \langle 1,1,1\rangle, \quad v_2 = \langle 1,-1,1\rangle, \quad v_3 = \langle 1,1,-1\rangle, \quad v_4 = \langle -1,-1,1\rangle$$

$$v_5 = \langle -1,1,1\rangle, \quad v_6 = \langle 1,-1,-1\rangle, \quad v_7 = \langle -1,1,-1\rangle, \quad v_8 = \langle -1,-1,-1\rangle,$$

which we think of geometrically as starting at the origin $O = (0,0,0)$.

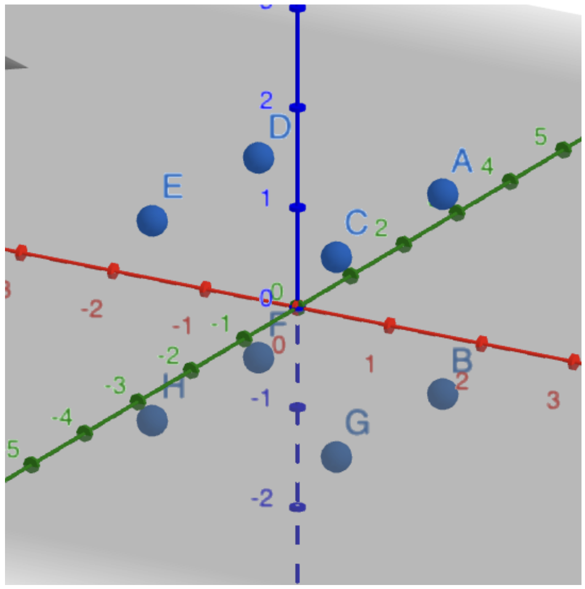

FIGURE 2. We plotted the tips of all the vectors listed here.

(a) Draw the endpoints of each of the 8 vectors v_i, $i = 1, \ldots, 8$.
Notice in the figure that we see all of the tips of the vectors are just points that form the corners of a cube.

(b) Show by direct computation that

$$v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 = \vec{0},$$

where $\vec{0} = (0, 0, 0)$ is the zero vector.
We can add all the components as follows:

$$\begin{aligned}
v_1 + v_2 &= \langle 1,1,1\rangle + \langle 1,-1,1\rangle = \langle 2,0,2\rangle \\
v_1 + v_2 + v_3 &= \langle 2,0,2\rangle + \langle 1,1,-1\rangle = \langle 3,1,1\rangle \\
v_1 + v_2 + v_3 + v_4 &= \langle 3,1,1\rangle + \langle -1,-,1,1\rangle = \langle 2,0,2\rangle \\
v_1 + v_2 + v_3 + v_4 + v_5 &= \langle 2,0,2\rangle + \langle -1,1,1\rangle = \langle 1,1,3\rangle \\
v_1 + v_2 + v_3 + v_4 + v_5 + v_6 &= \langle 1,1,3\rangle + \langle 1,-1,-1\rangle = \langle 2,0,2\rangle \\
v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 &= \langle 2,0,2\rangle + \langle -1,1-1\rangle = \langle 1,1,1\rangle \\
v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 &= \langle 1,1,1\rangle + \langle -1,-1,-1\rangle = \langle 0,0,0\rangle.
\end{aligned}$$

Thus, we can see that if we add the vectors one by one component-wise, we get the sum is $\vec{0}$, or the zero vector.

(c) Justify geometrically that

$$v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 = \vec{0},$$

by thinking about these vectors as arrows going around a cube.
Notice that if we draw out the vectors in the way we did in Figure 3, we can see that we have traced out a cube. If we add all these vectors together like this, we haven't gone anywhere, we end up back in the exact same spot on the cube. This is why we can add the vectors up and get $\vec{0}$, because if we think of the us going along the vectors as "traveling" along the cube, we end up in the same spot.

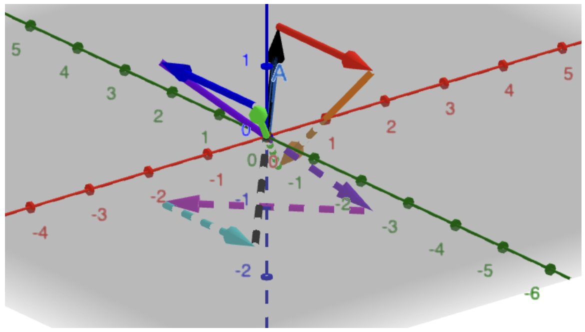

FIGURE 3. Here we plot the vectors as if when we are tracing from one vector to the next.