MAT 21C: PRACTICE PROBLEMS LECTURE 5

PROFESSOR CASALS (SECTIONS B01-08)

ABSTRACT. Practice problems for the fifth lecture of Part II, delivered May 10 2023. Solutions will be posted within 48h of these problems being posted.

Problem 1. Consider the vector $v = \langle -2, 1, -1 \rangle$.

(a) Compute the length $|v|$.
(b) Find the dot product $v \cdot v$.
(c) Verify that, in this case, $|v|^2 = v \cdot v$.
(d) Find the unit length vector $\frac{v}{|v|}$.

(a) In general, if $v = \langle v_1, v_2, v_3 \rangle$, then $|v| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. This gives

$$|v| = \sqrt{(-2)^2 + 1^2 + (-1)^2} = \sqrt{6}.$$

(b) The dot product formula for $u \cdot v$, where $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$, is

$$u \cdot v = u_1 \cdot v_1 + u_2 \cdot v_2 + u_3 \cdot v_3.$$

This gives

$$v \cdot v = (-2) \cdot (-2) + 1 \cdot 1 + (-1) \cdot (-1) = 6.$$

(c) Direct computation using results from (a),(b) gives

$$|v|^2 = (\sqrt{6})^2 = 6 = v \cdot v.$$

(d) Since $|v|$ is a scalar, dividing each component of v by $|v|$ gives

$$\frac{v}{|v|} = \left\langle \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right\rangle.$$

Problem 2. Consider the vectors $v = \langle 3, -7, 1 \rangle$ and $w = \langle -1, 4, 2 \rangle$.

(a) Find the dot product $u \cdot v$.
(b) Compute the length $|v|$.
(c) Compute the scalar component of u in the direction of v.
(d) Find the projection of u in the direction of v.
(e) What is the angle between u and v?

(Assuming $w = u$.)

(a) Using the same formula as stated in Problem 1(b), this gives

$$u \cdot v = (-1) \cdot 3 + 4 \cdot (-7) + 2 \cdot 1 = -3 - 28 + 2 = -29.$$

(b) Using the same formula as in Problem 1(a), this gives

$$|v| = \sqrt{3^2 + (-7)^2 + 1^2} = \sqrt{59}.$$

(c) The formula for the scalar component of u in the direction of v is

$$|u|cos\theta = \frac{u \cdot v}{|v|}.$$

where θ is the angle between u and v. This gives

$$\text{the scalar component of } u \text{ in the direction of } v \text{ equals } \frac{u \cdot v}{|v|} = \frac{-29}{\sqrt{59}}.$$

(d) The formula for the projection of u in the direction of v is

$$proj_v u = \left(\frac{u \cdot v}{|v|^2}\right) v = \left(\frac{u \cdot v}{v \cdot v}\right) v.$$

Hence,

$$proj_v u = \left(\frac{u \cdot v}{|v|^2}\right) v = \frac{-29}{(\sqrt{59})^2}\langle 3, -7, 1\rangle = \left\langle \frac{-87}{59}, \frac{203}{59}, \frac{-29}{59} \right\rangle.$$

(e) Let the angle between u, v is θ, then the formula for finding θ is

$$\theta = \arccos\left(\frac{u \cdot v}{|u||v|}\right).$$

Firstly, $|u| = \sqrt{(-1)^2 + 4^2 + 2^2} = \sqrt{21}$, so that we can find θ as follows:

$$\theta = \arccos\left(\frac{u \cdot v}{|u||v|}\right) = \arccos\left(\frac{-29}{\sqrt{21}\sqrt{59}}\right).$$

Problem 3. Consider a vector $v = (v_1, v_2, v_3)$.

(a) Show by direct computation that $|v|^2$ equals $v \cdot v$.
(b) As a second method, use the dot product-angle formula to deduce $|v|^2 = v \cdot v$.
(c) As a third method, justify geometrically in terms of projections why $|v|^2 = v \cdot v$.

(a) Using the same formula as stated in Problem 1(a), we have

$$|v|^2 = \left(\sqrt{v_1^2 + v_2^2 + v_3^2}\right)^2 = v_1^2 + v_2^2 + v_3^2 = v_1 \cdot v_1 + v_2 \cdot v_2 + v_3 \cdot v_3 = v \cdot v.$$

(b) Let θ be the angle between u, v, then the dot product formula can also be stated as:

$$u \cdot v = |u||v|cos\theta.$$

Since the angle between v and v is 0, then

$$v \cdot v = |v||v|cos(0) = |v|^2.$$

(c) Using the formula as stated in Problem 2(c), and note that the angle between v and v is 0. This implies the scalar component of v in the direction of v is

$$|v|cos(0) = \frac{v \cdot v}{|v|} \Longrightarrow |v|^2 = v \cdot v.$$

Problem 4. Consider the vectors $u = \langle -1, 2, 0\rangle$ and $v = \langle 3, 2, -5\rangle$.

(a) Show that $w = \langle 0, 0, 1\rangle$ is perpendicular to u but not to v.
(b) Show that $w = \langle 1, 1, 1\rangle$ is perpendicular to v but not to u.
(c) Show that $w = \langle 10, 5, 8\rangle$ is perpendicular to both v and u.

Vector u is perpendicular to v if and only if $u \cdot v = 0$.

(a) Since $w \cdot u = 0 \cdot (-1) + 0 \cdot 2 + 1 \cdot 0 = 0$ and $w \cdot v = 0 \cdot 3 + 0 \cdot 2 + 1 \cdot (-5) = -5 \neq 0$, this implies w is perpendicular to u but not v.

(b) Since $w \cdot u = 1 \cdot (-1) + 1 \cdot 2 + 1 \cdot 0 = 1 \neq 0$ and $w \cdot v = 1 \cdot 3 + 1 \cdot 2 + 1 \cdot (-5) = 0$, this implies w is perpendicular to v but not u.

(c) Since $w \cdot u = 10 \cdot (-1) + 5 \cdot 2 + 8 \cdot 0 = 0$ and $w \cdot v = 10 \cdot 3 + 5 \cdot 2 + 8 \cdot (-5) = 0$, this implies w is perpendicular to u and v.

Problem 5. Consider the vectors $u = \langle -5, 8, 1\rangle$ and $v = \langle -2, 3, 7\rangle$.

(a) Find a non-zero vector which is perpendicular to u but not v.
(b) Find a non-zero vector which is perpendicular to v but not u.
(c) Find a non-zero vector which is perpendicular to both u and v.

Vector u is perpendicular to v if and only if $u \cdot v = 0$.

(a) Let $w = \langle 2, 1, 2\rangle$, then

$$w \cdot u = 2 \cdot (-5) + 1 \cdot 8 + 2 \cdot 1 = 0,$$

and

$$w \cdot v = 2 \cdot (-2) + 1 \cdot 3 + 2 \cdot 7 = 13.$$

This gives w is perpendicular to u but not v.

(b) Let $w = \langle 5, 1, 1\rangle$, then

$$w \cdot u = 5 \cdot (-5) + 1 \cdot 8 + 1 \cdot 1 = -16,$$

and

$$w \cdot v = 5 \cdot (-2) + 1 \cdot 3 + 1 \cdot 7 = 0.$$

This gives w is perpendicular to v but not u.

(c) Suppose $w = \langle x, y, z\rangle$, then perpendicular to both u, v implies

$$-5x + 8y + z = 0, \text{ and } -2x + 3y + 7z = 0.$$

Set $z = 1$, then

$$-5x + 8y + 1 = 0, \text{ and } -2x + 3y + 7 = 0,$$

and by solving these two functions gives $x = 53, y = 33$. Hence, let $w = \langle 53, 33, 1\rangle$, then

$$w \cdot u = -265 + 264 + 1 = 0, \ w \cdot v = -106 + 99 + 7 = 0.$$

This gives w is perpendicular to u and v.

Problem 6. Consider the plane $\pi = \{ax + by + cz = 0\}$ for some $a, b, c \in \mathbb{R}$. Explain using the dot product why $\langle a, b, c\rangle$ is the perpendicular direction to π.

Hint: The left hand side $ax + by + cz = 0$ of the equation can be writtent as $\langle a, b, c\rangle \cdot \langle x, y, z\rangle$. Also, the endpoint of $\langle x, y, z\rangle$ belong to π if and only if $ax + by + cz = 0$.

Consider two arbitrary points of the plane, with

$$(x_1, y_1, z_1), (x_2, y_2, z_2) \in \pi.$$

Definition of π gives

$$ax_1 + by_1 + cz_1 = 0, \quad ax_2 + by_2 + cz_2 = 0,$$

which implies

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0,$$

and it is equivalent to

$$\langle a, b, c\rangle \cdot \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle = 0.$$

Since $\langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$ represents the vector that connects arbitrary two points in the plane, we have $\langle a, b, c\rangle$ perpendicular to vectors that connects arbitrary two points in the plane. Hence, $\langle a, b, c\rangle$ is the perpendicular direction to π.